Introduction to Electric Power Systems
Lecture 8
Maximum Power Transfer and Voltage Stability

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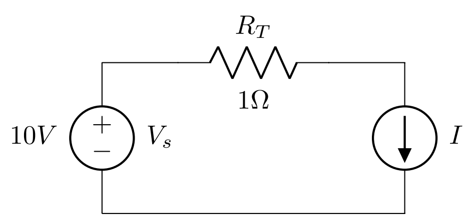
Contents

1 Maximum Power Transfer In DC Circuits

First we will consider DC circuits in order to develop intuition for maximum power transfer. Consider the following DC circuit in which a DC voltage source supplies power to a load modeled as a current source, through a transmission line with impedance R_T:



For the following question, you may find these equations helpful. The second equation uses the KVL relationship $V_L = V_S - IR_T$.

$$P_L = V_L I \tag{1}$$
$$= IV_S - I^2 R_T \tag{2}$$

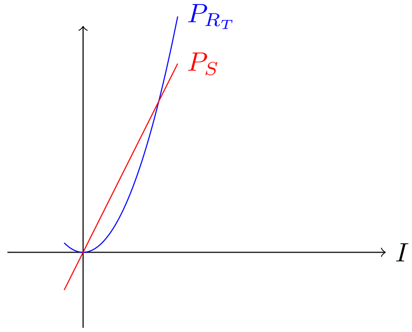
Q. *Plot P_L vs I with P_L on the vertical axis and I on the horizontal axis*

Q. *What current I delivers the maximum power to the load?*

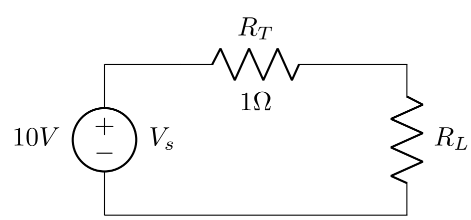
There are two ways to interpret this result.

1. From equation (1) we see that power is the product of voltage and current. To get the max amount of power, the current cannot make the voltage at the load drop too much. We see that the optimal current makes $V_L = \frac{V_S}{2}$.

2. Equation (2) can be interpreted as: (Load Power) = (Source Power) - (Transmission Losses). From this equation it is clear that, while the source power IV_S scales linearly with current with slope V_S, the transmission losses scale with the *square* of the current. So initially the power drawn from the source for each marginal unit of current is greater than the power lost due to each marginal unit of current. But eventually, because losses go with I^2, the losses will be *greater* than the power drawn from the source for each marginal unit of current.



While we're at it, consider this second circuit in which a DC voltage source supplies power to a constant resistance load with resistance R_L, through a transmission line with resistance R_T:



Q. *What resistance R_L delivers the maximum power to the load?*

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2 Maximum Power Transfer In AC Circuits

We looked at DC circuits only to develop intuition, the AC circuit equations are really what we are interested in. AC transmission lines have limits for both the amount of real and reactive powers that can be transmitted across the line for a given sending voltage magnitude. The limits on P_R and Q_R can be expressed in terms of the receiving end voltage V_R (or sending end voltage V_S), holding the angle separation δ constant, or in terms of δ, holding V_R and V_S constant.

Note, these derivations consider both the source and the load as voltage sources, which differs from the DC circuit derivations above which modeled the load as a current source or constant impedance.

First we hold voltage magnitudes constant and determine the voltage angle δ that results in maximum power transfer for a lossless line. Then we consider a lossy line (a line with resistance as well as reactance). Similar concepts apply to determining the voltage magnitude that results in the maximum power transfer, however we do not cover that here.

2.1 Lossless lines

First, we focus on the maximum P_R that can be transferred to a load across a purely reactive (lossless) line. The shunt admittances are included as well.

$$\begin{aligned} I_R &= \frac{V_S - V_R}{Z'} - \frac{Y'}{2}V_R \\ &= \frac{V_S e^{j\delta} - V_R}{jX'} - \frac{j\omega C'l}{2}V_R \end{aligned} \tag{5.4.24}$$

and the complex power S_R delivered to the receiving end is

$$\begin{aligned} S_R = V_R I_R^* &= V_R\left(\frac{V_S e^{j\delta} - V_R}{jX'}\right)^* + \frac{j\omega C'l}{2}V_R^2 \\ &= V_R\left(\frac{V_S e^{-j\delta} - V_R}{-jX'}\right) + \frac{j\omega Cl}{2}V_R^2 \\ &= \frac{jV_R V_S\cos\delta + V_R V_S\sin\delta - jV_R^2}{X'} + \frac{j\omega Cl}{2}V_R^2 \end{aligned} \tag{5.4.25}$$

Q. *What is the equation for P_R?*

Q. *Plot P_R vs δ, with P_R on the vertical axis and δ on the horizontal axis.*

Q. *What is $P_{R,\max}$?*

Q. *What δ gives $P_{R,\max}$?*

This equation can also be expressed in terms of the per unit voltages, the surge impedance load Z_c, the wavelength of traveling wave λ (p. 276 GOS), and the length of the line l:

$$P = \frac{V_S V_R \sin\delta}{Z_c \sin\beta l} = \left(\frac{V_S V_R}{Z_c}\right)\frac{\sin\delta}{\sin\left(\frac{2\pi l}{\lambda}\right)} \tag{5.4.28}$$

Expressing V_S and V_R in per-unit of rated line voltage,

$$P = \left(\frac{V_S}{V_{\text{rated}}}\right)\left(\frac{V_R}{V_{\text{rated}}}\right)\left(\frac{V_{\text{rated}}^2}{Z_c}\right)\frac{\sin\delta}{\sin\left(\frac{2\pi l}{\lambda}\right)}$$

$$= V_{S\text{p.u.}} V_{R\text{p.u.}} (\text{SIL}) \frac{\sin\delta}{\sin\left(\frac{2\pi l}{\lambda}\right)} \quad \text{W} \tag{5.4.29}$$

The "surge impedance load" (SIL or Z_c) is just a (real valued) property of a lossless line defined as $\sqrt{\frac{L}{C}}$. For a given line with inductance/shunt capacitance ration $\frac{L}{C}$, if a resistor is placed on the end of the line with magnitude Z_c, the voltage at the load will be equal to the voltage at the source. It's not immediately obvious how that relates to maximum power transfer across a line, other than that Z_c is a parameter of the line.

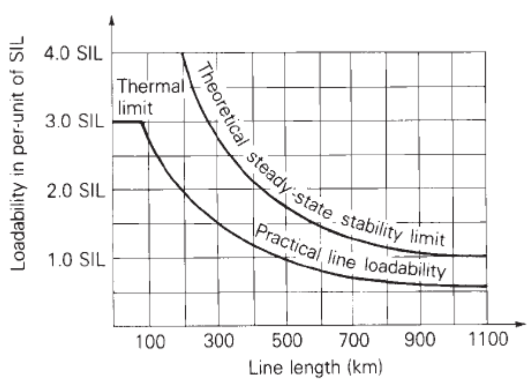
Properties of $P_{R,\max}$ that are evident from the SIL formulation:

1. Increases with voltage squared
2. Decreases with line line length

These make sense intuitively. Power goes with voltage squared, and increasing the line length will increase the impedance of the line.

FIGURE 5.12

Transmission-line loadability curve for 60-Hz overhead lines—no series or shunt compensation



2.2 Lossy lines

The maximum power flow can also be derived for lines with series resistance:

$$A = \cosh(\gamma\ell) = \mathrm{A}\underline{/\theta_A}$$
$$B = Z' = Z'\underline{/\theta_Z}$$
$$V_S = \mathrm{V_S}\underline{/\delta} \qquad V_R = \mathrm{V_R}\underline{/0^\circ}$$

Solving (5.2.33) for the receiving-end current,

$$I_R = \frac{V_S - AV_R}{B} = \frac{\mathrm{V_S}e^{j\delta} - \mathrm{AV_R}e^{j\theta_A}}{Z'e^{j\theta_Z}} \tag{5.5.1}$$

The complex power delivered to the receiving end is

$$S_R = \mathrm{P_R} + j\mathrm{Q_R} = V_R I_R^* = \mathrm{V_R}\left[\frac{\mathrm{V_S}e^{j(\delta-\theta_Z)} - \mathrm{AV_R}e^{j(\theta_A-\theta_Z)}}{Z'}\right]^*$$
$$= \frac{\mathrm{V_R V_S}}{Z'}e^{j(\theta_Z-\delta)} - \frac{\mathrm{AV_R^2}}{Z'}e^{j(\theta_Z-\theta_A)} \tag{5.5.2}$$

The real and reactive power delivered to the receiving end are thus

$$\mathrm{P_R} = \mathrm{Re}(S_R) = \frac{\mathrm{V_R V_S}}{Z'}\cos(\theta_Z - \delta) - \frac{\mathrm{AV_R^2}}{Z'}\cos(\theta_Z - \theta_A) \tag{5.5.3}$$

$$\mathrm{Q_R} = \mathrm{Im}(S_R) = \frac{\mathrm{V_R V_S}}{Z'}\sin(\theta_Z - \delta) - \frac{\mathrm{AV_R^2}}{Z'}\sin(\theta_Z - \theta_A) \tag{5.5.4}$$

Q. *Does the equation for P_R agree with the lossless line equation for P_R? ($\theta_A = 0$ and $\theta_Z = 90°$)*

Q. *What δ gives the max real power transfer for a lossy line? Is it $90°$?*

Q. *What is $P_{R,\max}$? How does it compare with $P_{R,\max}$ for the lossless line?*

Q. *Intuitively, how do the results for the AC circuit relate to the results for the constant current DC circuit we started with? (Think in terms of line losses and power generated)*

Q. *Intuitively, how do the results for the AC circuit relate to the results for the constant impedance DC circuit? What AC impedance maximizes the power consumed by the load?*

A *Bonus Material*: Power Equations and The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra, which guarantees n complex-valued roots for a polynomial of degree n, guarantees complex solutions to Eqn. (2) for any P_L. But (2) is for a DC circuit and thus real-valued solutions are required. The Fundamental Theorem of Algebra doesn't tell us when there are real-valued solutions to (2).

The AC circuit analogue to (2) is:

$$S_L = V_L I^* = V_S I^* - I I^* Z_T \tag{3}$$

For AC circuits, complex-valued solutions *are* viable. However, unlike (2), (3) is not a polynomial. The Fundamental Theorem of Algebra guarantees complex-valued solutions for polynomials only. So the Fundamental Theorem of Algebra apply to (3) and doesn't help power system engineers much, or at least not in this context.