Introduction to Electric Power Systems
Lecture 2
Complex Power

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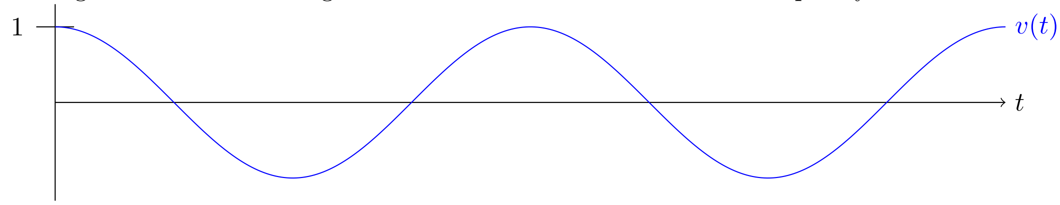
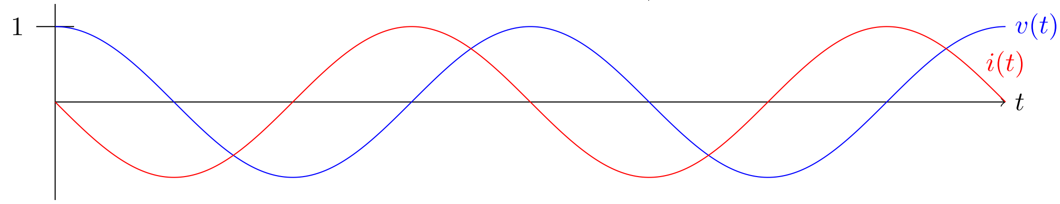
Contents

1 Leading and Lagging Current

The voltage on an AC electric grid is a sinusoid at the fundamental frequency.



If a network is constructed out of linear circuit elements, then the current wave will be a sinusoid as well.



Q. *Is the current wave above leading or lagging? Is the current created by a capacitor or an inductor?*

Q. *Draw the corresponding phasor diagram for the capacitor circuit:*

Q. *Draw the current for a circuit with a 1 Ω inductor:*

Q. *Draw the corresponding phasor diagram for the inductor circuit:*

Q. *And for a circuit with a 1 Ω resistor:*

Q. *Draw the corresponding phasor diagram for the resistor circuit:*

2 Instantaneous Power

Q. *How can AC power deliver positive instantaneous power when the load voltage is negative?:*

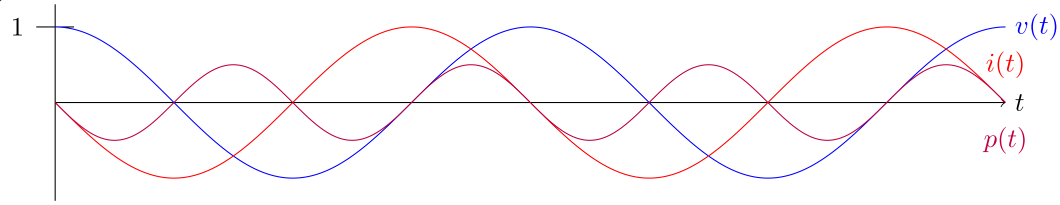
Q. *In AC circuits, electrons move back and forth with the AC current. So electrons are not delivered to either side of the load, on average. Despite this, generators still deliver power to the load. How do generators send power without delivering electrons on average?:*

Instantaneous power is the product of voltage and current at any moment in time: $p(t) = v(t)i(t)$. If $v(t)$ and $i(t)$ are sinusoids, then $p(t)$ will be a sinusoid as well. However even if $v(t)$ and $i(t)$ are both centered at zero, if $v(t)$ and $i(t)$ are not in phase, $p(t)$ will have a nonzero vertical offset/average value. The average value of the $p(t)$ wave is the average power that is delivered to the element for which $v(t)$ and $i(t)$ are defined.

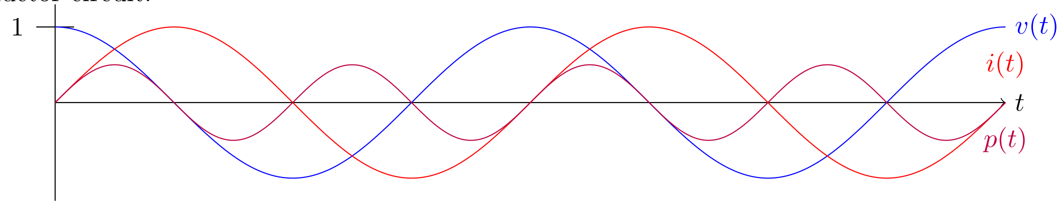
In addition to the average value of $p(t)$, the amplitude of $p(t)$ is also important because the amplitude of $p(t)$ is necessary for determining the amplitude of $i(t)$, and $i(t)$ determines the line losses on the network. The line losses quantify how much real power is converted to heat when transmitting power from one location on the grid to another. Line losses are undesirable both because they are wasteful and because they increase the temperature of transmission lines. As the transmission line temperature increases, the lines expand and sag more. Transmission constraints are determined by how much a transmission line can expand/sag.

Multiplying the voltage and the current values at each instant gives the power wave for the capacitor, inductor, and resistor circuits described above:

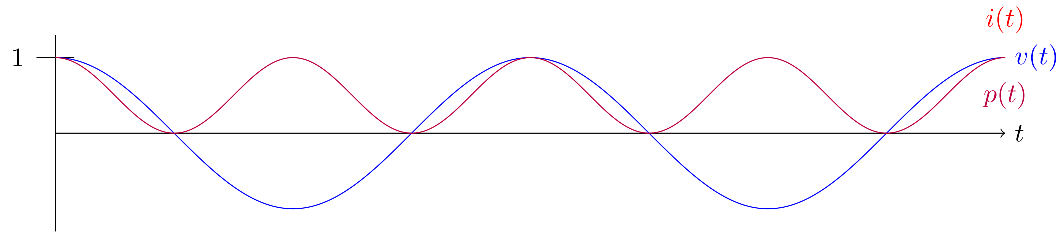
Capacitor circuit:



Inductor circuit:



Resistor circuit (the current wave is under the voltage wave):



Q. *What is the frequency of the instantaneous power waves relative to the frequency of the voltage and current waves?:*

Q. *What is the average value of each of the instantaneous power waves?:*

Voltage and current waves at frequency ω with arbitrary amplitudes $|V_{\max}|$ and $|I_{\max}|$ and phaseshifts α and β are defined by the following equations:

$$v(t) = |V_{\max}| cos(\omega t + \alpha)$$
$$i(t) = |I_{\max}| cos(\omega t + \beta)$$

These waves can be expressed as phasors using the RMS representations of amplitude (RMS is assumed in power systems when there is no subscript):

$$v(t) \leftrightarrow V = |V|\angle\alpha, |V| = \frac{|V_{\max}|}{\sqrt{2}}$$
$$i(t) \leftrightarrow I = |I|\angle\beta, |I| = \frac{|I_{\max}|}{\sqrt{2}}$$

Instantaneous power is calculated by multiplying $v(t)$ and $i(t)$. Note, instantaneous power cannot be calculated from phasors, because the instantaneous portion of the complex exponential $e^{j\omega t}$ is discarded in the phasor derivation. Using some convenient trigonometric identities:

$$2\cos(x)\cos(y) = \cos(x - y) + \cos(x + y) \quad (1)$$
$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y) \quad (2)$$

we can describe the instantaneous power as the sum of two sinusoids:

$$\begin{aligned} p(t) &= v(t)i(t) \\ &= |V_{\max}| cos(\omega t + \alpha) * |I_{\max}| cos(\omega t + \beta) \\ &= \frac{1}{2}|V_{\max}||I_{\max}|[\cos(\alpha - \beta) + \cos\big(2(\omega t + \alpha) - (\alpha - \beta)\big)] \\ &= |V||I|[\cos(\alpha - \beta) + \cos(\alpha - \beta)\cos\big(2(\omega t + \alpha)\big) + \sin(\alpha - \beta)\sin\big(2(\omega t + \alpha)\big)] \\ &= |V||I|\cos(\alpha - \beta)[1 + \cos\big(2(\omega t + \alpha)\big)] + |V||I|\sin(\alpha - \beta)[\sin\big(2(\omega t + \alpha)\big)] \end{aligned}$$

Q. *What is the average value of the first term?:*

Q. *What is the average value of the second term?:*

Q. *What is the average value of the instantaneous power p(t)?:*

Q. *What is the amplitude of the sinusoid in the first term?:*

Q. *What is the amplitude of the sinusoid in the second term?:*

Q. *What is the amplitude of the instantaneous power p(t) sinusoid? Hint:*

$$a\cos(x) + b\sin(x) = \sqrt{a^2+b^2}\sin\left(x + \arctan(\frac{a}{b})\right) \tag{3}$$

3 Mathematical and Visual Description of Real and Reactive Power

$$p(t) = \underbrace{|V||I|\cos(\alpha-\beta)}_{P}[1+\cos\left(2(\omega t+\alpha)\right)] + \underbrace{|V||I|\sin(\alpha-\beta)}_{Q}[\sin\left(2(\omega t+\alpha)\right)] \tag{4}$$

"**Real power**", denoted by **P** in power systems analysis, is defined as the average power delivered, and is given by the first term in the instantaneous power equation, $|V||I|\cos(\alpha-\beta)$. Note, this average power is also equal to the amplitude of the real power wave, which will always be centered at $|V||I|\cos(\alpha-\beta)$ and touch 0.

"**Reactive power**", denoted by **Q** in power systems analysis, is defined as the amplitude of the second term in the instantaneous power equation, $|V||I|\sin(\alpha-\beta)$. Reactive power results in net zero power delivery, but does increase the current flow on the network.

"**Apparent power**", denoted by **S** in power systems analysis, describes the instantaneous power wave (4). Mathematically, S is the complex sum of the real and reactive power, $S = P + Qj$. Apparent power can be calculated from V and I using the complex power formula:

$$S = VI^*$$

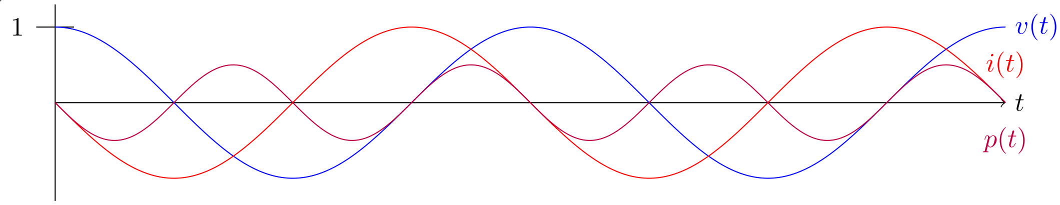
Real power is the real portion of S, and reactive power is the imaginary portion of S.

Q. *Why does the equation for power use the complex conjugate of the current phasor?:*

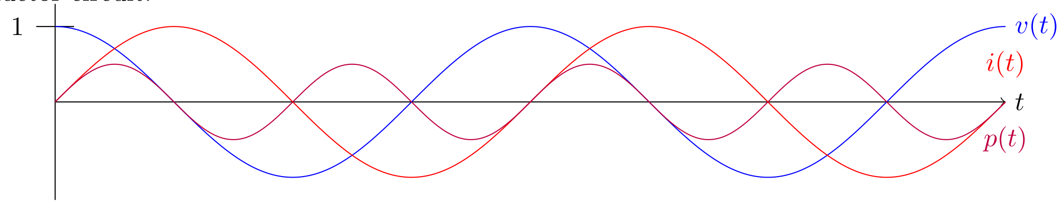
Q. *Draw three complex planes. On one, draw the complex impedances for a resistor, a capacitor, and an inductor. On the second, draw the current phasors for a resistor circuit, a capacitor circuit, and an inductor circuit. On the third, draw the apparent power for each of the three circuits.*

Returning to the time-domain plots for the three basic circuit examples:

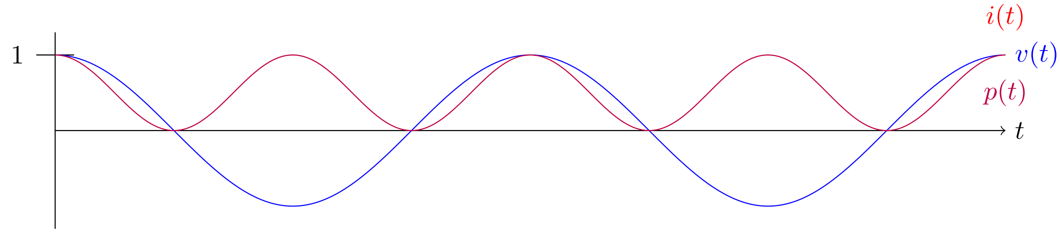
Capacitor circuit:



Inductor circuit:



Resistor circuit (the current wave is under the voltage wave):

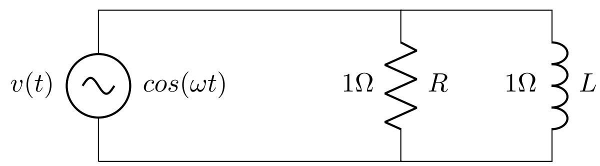


Q. *What is the real power and reactive power for each circuit?:*

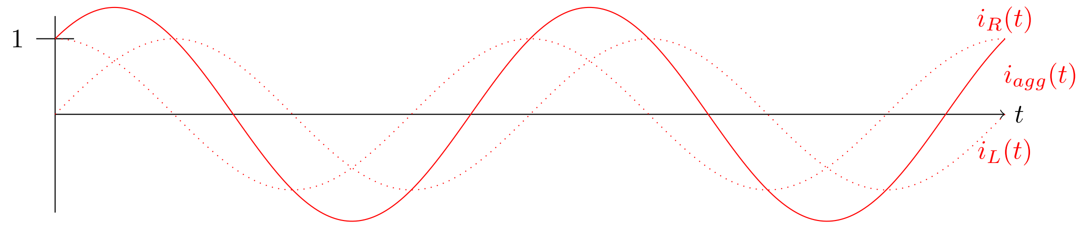
4 Circuit Examples

4.1 Parallel Circuit

The following circuit has a voltage source supplying power to a load that consists of a resistor and an inductor in parallel:



The current and power waves for the resistor and inductor individually are given by the plots above. (Both the resistor and inductor see the source voltage because they are in parallel). Because the components are in parallel, the aggregate current wave is given by adding the current waves using the trigonometric identity (3).

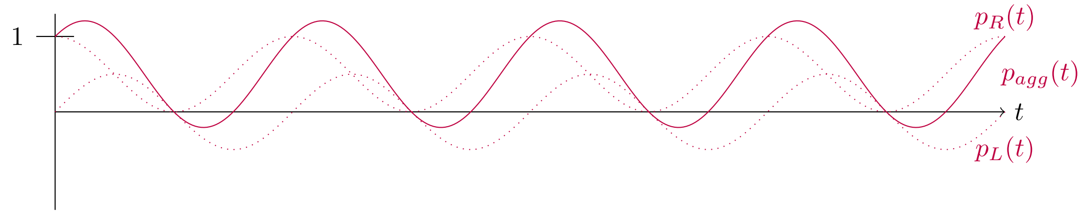


Q. *What is the amplitude of the aggregate current wave? And the RMS amplitude?*

Q. *Does the inductor current deliver any real power?*

Q. *Is the peak amplitude of the aggregate current wave affected by the inductor current?*

The aggregate power wave is also given by adding the power waves for each component (though instantaneous power waves add for any circuit configuration, not just for loads in parallel).



Q. *Label the above plot with S, P, and Q.*

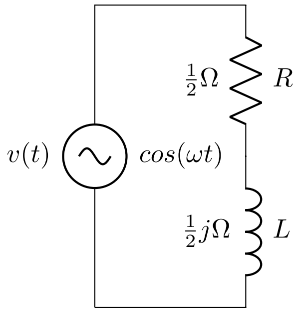
Q. *Calculate the amplitude of the power wave $|S|$ using the trigonometric identity (3). Just the amplitude of the wave, don't worry about the vertical offset or phaseshift.*

Q. *Calculate the amplitude of the power wave using phasors.* $V = \frac{1}{\sqrt{2}}\angle 0$ *and* $I = 1\angle\frac{-\pi}{4}$

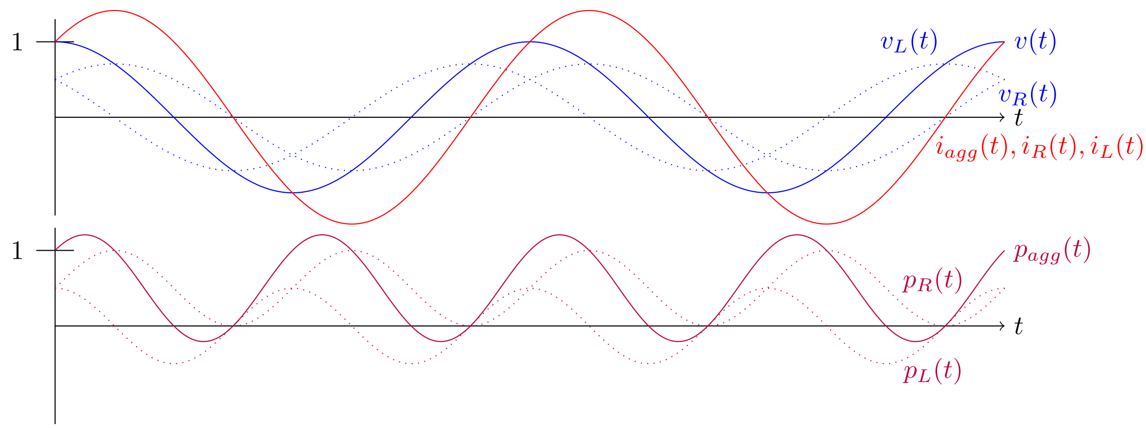
4.2 Series Circuit

For parallel circuits, it is easy to draw the power waves for each individual component because each component has voltage V across it. For series circuits, it is more difficult to draw the power waves for each component because the voltage across each component will have a phase shift (thus the power in each component will be phase shifted as well). But the result is still the same—the aggregate power wave can be expressed as the sum of a real power wave with average value P, and a reactive power wave with average value 0.

Series circuit with an equivalent impedance to the parallel circuit above:



Voltage, current and power waves for the series circuit:



5 Conceptual Description of Real and Reactive Power

Any current wave can be decomposed into a current wave that is in phase with the corresponding voltage wave, and a current wave that is orthogonal (ninety degrees out of phase) with the voltage wave using the trigonometric identity (1).

Reactive power is the power that oscillates back and forth due to the current wave that is orthogonal (90 degrees shifted) to the voltage wave. Because this current wave is ninety degrees out of phase with the voltage, and power is the product of voltage and current, the reactive power wave will have an average value of 0.

Real power is the power that oscillates back and forth due to the portion of the current wave that is in phase with the voltage wave. Because the waves are in phase, and the real power wave touches zero each cycle, the average value of the power wave will also be equal to P.

Apparent power is the sum of the orthogonal real and reactive power waves, and describes the power of the aggregate current wave (which is equal to the sum of the in phase current wave and the orthogonal current wave). Because the reactive power wave has an average power of zero, the average of the apparent power is P.

5.1 Properties of Apparent Power

- $S = P + Qj = VI^*$
- S contains the information for the instantaneous power wave, which is a wave with twice the frequency of the voltage and current waves and a vertical offset. It is a small abuse of notation to put the complex S vector on the same complex plane as V, I, and Z (which is defined for phasors with frequency ω or constants), however putting S on the same complex plane is still generally done because it is convenient.
- $|S| = \sqrt{P^2 + Q^2}$ is the peak amplitude of the instantaneous power wave, not an RMS quantity.

5.2 Properties of Reactive Power

- Reactive power is measured in **Volt-Ampere Reactive (VARs)**. A VAR is the same size as a Watt, but the different terminology allows us to keep track of them separately. This is a semantic distinction that represents the physical separation between real and reactive power.
- Reactive power, like real power, is conserved across the system.

Q. *Why do we care where reactive power is generated?*
