Introduction to Electric Power Systems
Lecture 10
The Admittance Matrix

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1 Power Flow Motivation

Up to this point, much of our focus has been single line networks (e.g. basic phasor addition), and angle stability analysis for single generators (Equal Area Criterion). However power systems are vast electric networks with many lines connecting many loads and generators. Furthermore, unlike communication networks, power cannot be routed from one location to another according to a specific path—once power is injected onto the grid, it flows according to the laws of physics and anyone with access to the grid can extract power. Grid operators must be able to describe how generation and load instances will affect the line flows and voltages on the network. Understanding the relationship between injections/extractions and line flows/voltages allows grid operators to answer questions such as

- If the load at a given node increases, will it result in an undervoltage at any node on the network?
- If the generation is increased at a given node, will it result in a lineflow that is too large on any line?

The power flow on a network is determined by two equations, which will explain in this chapter:

1. Vector-valued Ohm's Law: $I = YV$
2. The power injection equation: $S = VI^*$

The equations can be combined to give the power flow equations for the network—a system of nonlinear equations that relate the nodal power injections to the network voltages. Before introducing the network-wide power flow equation, we introduce the bus admittance matrix.

2 Bus Admittance Matrix

The bus admittance matrix Y_{bus} allows us to write Ohms law for a network of any size in a single line: $I = Y_{\text{bus}}V$. Often the "bus" subscript is omitted when it is obvious from the context that Y is a bus admittance matrix. We will use the Y shorthand. Y contains *both* the network connectivity and the impedance information of the network, in one intuitively understood matrix. Before we get into specifics, some definitions:

- "*Bus*": a point in the network at which a generator, load, shunt admittance or transformer is connected, or three or more lines come together. The bus is assumed to have a uniform voltage (all devices/lines attached to the bus are attached to this uniform voltage). Bus is often used synonymously with "node."
- "*Branch*": a connection between nodes, usually a power line. Branch is used synonymously with "line" and "edge."
- "*Network*": a set of buses and branches, usually connected. A network can be defined as a subsection of a grid, i.e. a distribution network is attached to the transmission network.
- "*Bus injection current*": the current that flows into a given bus, either from a bus outside of the network or a load/generator. Bus injection current does *not* include the current through a shunt admittance.
- "*Shunt admittance current*": The current that flows from one bus to ground through the shunt admittance.
- "*Branch flow current*": The current that flows from one bus to another bus across a branch.

The complex-valued nodal admittance matrix Y relates the vector of complex voltages at each bus to the vector of current *injections* at each bus:

$$I = YV \tag{1}$$

Note also that the vector I represents the bus injection current in this context. If $Y(i,k)$ is the $(i,k)^{\text{th}}$ entry of Y, the admittance of a line between nodes i and k is y_{ik}, and the shunt admittance at node i is y_i, Y is constructed using the following rules:

$$Y(i,k) = \begin{cases} y_i + \sum_{l \neq i} y_{il}, & \text{if} \quad i = k \\ -y_{ik}, & \text{if} \quad i \neq k \end{cases} \tag{2}$$

An important subtlety is that the current injections in the I vector are in *parallel* with the current flowing through the shunt admittance y_i.

Q. *What is value of the $(i,k)^{\text{th}}$ entry of Y when the i and k nodes aren't connected? Is this what you would expect?*

Q. *What is the value of y_i when no shunt admittance is attached to bus i? Is this what you would expect?*

The structure for Y comes from a combination of Ohms law and KCL. To develop intuition for why Y is constructed using the rules in (2), it is helpful to think of Y for a 2 bus system in which the two buses are attached by a line with admittance y_{12}, with no shunt admittances:

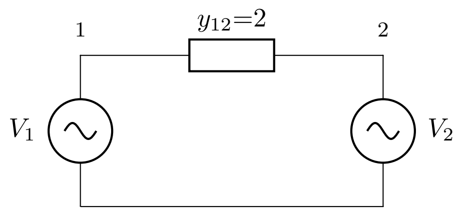

Figure 1: 2 Node Example Circuit

Q. *What is Y for this network?*

Q. *What is the expression for the current injection at bus 1?*

Q. *How does the structure of Y implictly apply ohms law?*

Now consider the 2 node network with a shunt admittance:

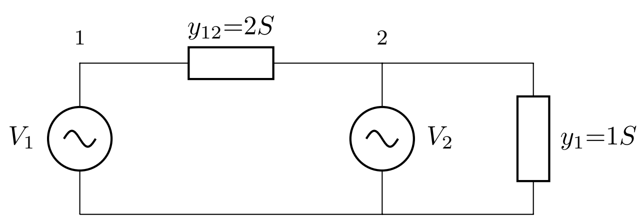

Figure 2: 2 Node Example Circuit with Shunt Admittance

Q. *What is Y for this network?*

Q. *Does the Ohms law intuition still apply to node 2 of this circuit?*

Now consider a 3 bus network with the following Y matrix:

$$Y = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 7 & -3 \\ 0 & -3 & 4 \end{bmatrix}$$

Q. *What do the 0s in the (3,1) and (1,3) entries imply?*

Q. *Which node(s) have shunt admittance(s)?*

Q. *Can you demonstrate that current balance holds for node 2 using $I = YV$?*

A *Bonus Material*: Nodal Impedance Matrix

Y can be inverted to get the nodal impedance matrix $Z = Y^{-1}$. Z is computed using a matrix inverse of Y, rather than directly, because unlike Y, Z is not sparse and cannot be constructed from intuitive rules. For these reasons, Z is less commonly used than Y. Given the bus current injections, Z gives the network voltages:

$$V = ZI$$

A small technical point: If Y does not have any shunt admittances then it will be rank deficient by 1. The eigenvalue of 0 will correspond to the nullspace eigenvector with 1 in each entry. When matrices have an eigenvalue of 0 then they are called "singular" and cannot be inverted. In this case, $\tilde{Z}$ can be calculated by taking the pseudoinverse of Y (the inverse ignoring the mode corresponding to the zero eigenvalue): $\tilde{Z} = Y^{\dagger}$. $\tilde{Z}$ can play the roll of Z for most applications.

Q. *Bonus question: Why does the eigenvector of all 1s correspond to the zero eigenvector of Y when Y does not have shunts?*

B *Bonus Material*: Graph Laplacians and the Laplace Operator

The nodal admittance matrix Y is actually an example of a graph Laplacian, which is the tool used for spacial discretization in computer science and other applied math fields. Matrices of this structure are the foundation of a field called "spectral graph theory."

In the literature, "Laplacian" is used to refer to both the continuous and discrete operator. At risk of some confusion, we will use "Laplacian Matrix" to refer to the discrete operator, and "Laplace Operator" to refer to the continuous operator in this section.

The Laplace operator occurs in differential equations that describe many physical phenomena, such as electric and gravitational potentials, the diffusion equation for heat and fluid flow, wave propagation, and quantum mechanics. The Laplace operator represents the flux density of the gradient flow of a function.

The Laplacian Matrix is the Laplace Operator for discretized space. Circuits are a canonical example of discretized space, if the usual assumptions are made (current flows only though circuit connections, wires have negligible or lumped impedance, etc.).

In vector calculus, divergence is a vector operator that operates on a vector field, producing a scalar field giving the quantity of the vector field's source at each point. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

The divergence operator on vector field $\mathbf{F}$ is $\operatorname{div}\mathbf{F}$:

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

The Laplace operator is the divergence of the gradient. If ϕ denotes the electrostatic potential associated to a charge distribution q, then the charge distribution itself is given by the negative of the Laplacian of ϕ:

$$q = -\varepsilon_0 \Delta\varphi$$

where ε_0 is the electric constant. This is a consequence of Gauss's law. Indeed, if V is any smooth region, then by Gauss's law the flux of the electrostatic field E is proportional to the charge enclosed:

$$\int_{\partial V} \mathbf{E} \cdot \mathbf{n}\, dS = \int_V \operatorname{div}\mathbf{E}\, dV = \frac{1}{\varepsilon_0} \int_V q\, dV$$

where the first equality is due to the [divergence theorem]. Since the electrostatic field is the (negative) gradient of the potential, this now gives

$$-\int_V \operatorname{div}(\operatorname{grad}\varphi)\, dV = \frac{1}{\varepsilon_0} \int_V q\, dV.$$

So, since this holds for all regions V, we must have

$$\operatorname{div}(\operatorname{grad}\varphi) = -\frac{1}{\varepsilon_0} q$$

To get a vector field from a voltage (charge distribution) field you take the gradient. To get the net flows into/out of any point in space you evaluate the Laplacian at that point.[1]

In the graph context that is exactly what the Laplacian admittance matrix does for an electric network—given a voltage distribution, Y gives the net injection/extraction at each node. The discrete structure is appropriate because the current only flows in/out at the nodes, and only along the branches.

[1] The same approach implies that the negative of the Laplacian of the gravitational potential is the mass distribution.