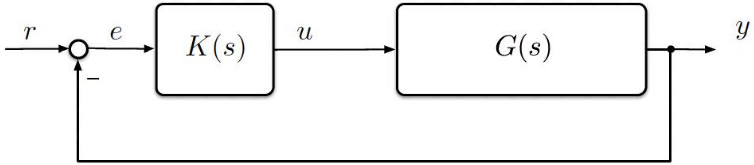
# Review:
# Closed Loop Stability & Design Specifications

Dr. Keith Moffat

# Classical control (Week 4)

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Desired: Good closed loop behavior (robust, etc.)

Approach: Choose K(s) with as few calculations as possible

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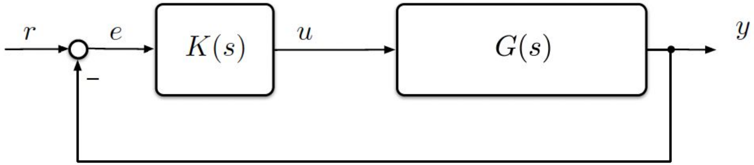


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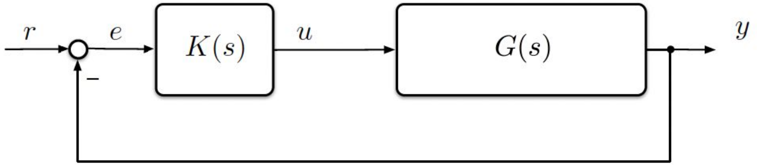


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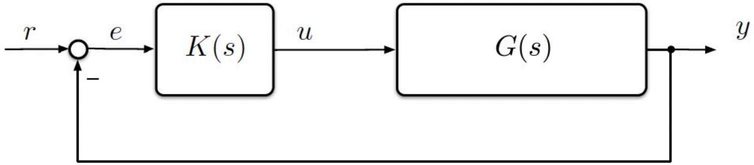


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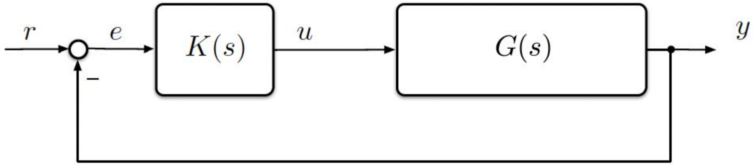


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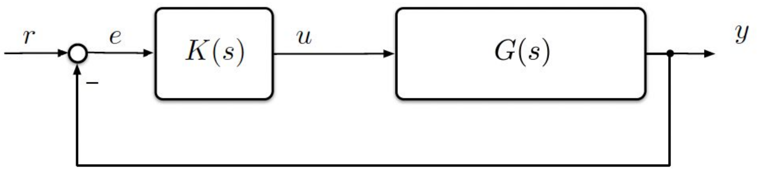
What do we want to be robust to?

- Model mismatch
- Delay
- Disturbances and measurement noise

# Robust closed loop stability

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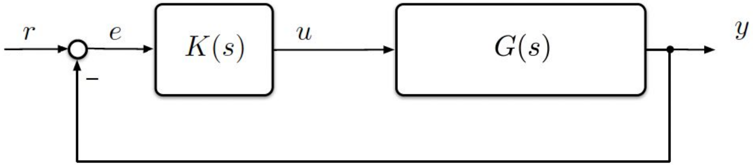


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To design robust controllers, we need some notion of how <u>close the closed loop system is to instability.</u>

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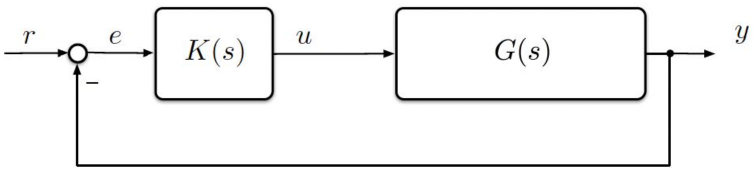


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The techniques we have covered

- Root Locus
- Nyquist
- Bode
- Small gain criterion

can be seen as robust design techniques.

# Bode plots (Week 3)

*Bode plots illustrate transfer functions*

Why is the plot in log-log axes?

Signal magnitude factors are additive in the log scale: $\lg(a \cdot b) = \lg a + \lg b$

Why are Bode plots plotted in "decibels" (i.e. scaled by 20)?

Because that's how they started doing it ~100 years ago

: 'bel' = signal energy in log-scale

$$|G|_{\mathsf{dB}} = 10 \lg |G|^2 = 20 \lg |G|$$

Helpful resource: https://lpsa.swarthmore.edu/Bode/BodeHow.html

Less-helpful resource:
https://www.youtube.com/watch?v=QHPRrUTn5vQ&ab_channel=CarlosBodefan

Bode:

Bode skiing:

# Bode plots

*Bode plots illustrate the transfer function/frequency response of a system*

# Frequency response

*The frequency response of a system is its magnitude and phase shift for every single-frequency input*

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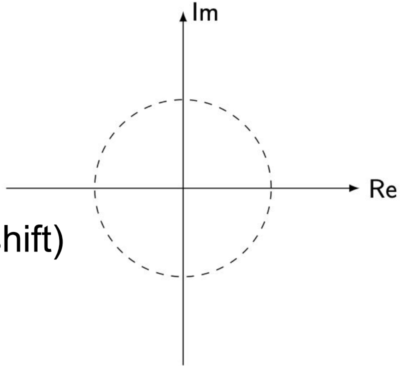
*or*

Magnitude and phase shift can be represented in a single complex plane plot (j = 90° shift)

→ Nyquist plots



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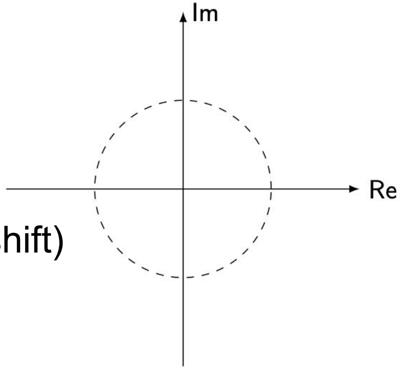



Magnitude and phase shift can be represented in a single complex plane plot (j = 90° shift)

→ Nyquist plots

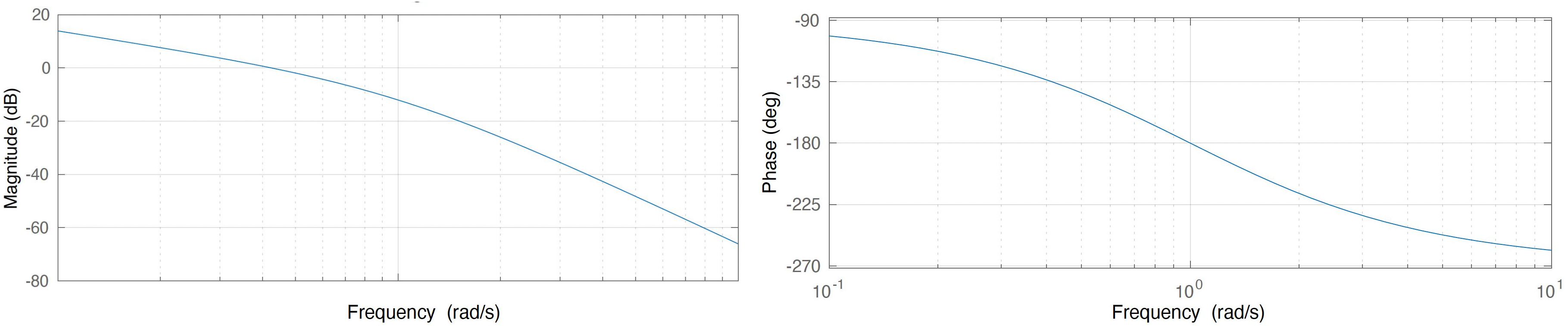
Replacing s in the transfer function with jw gives you the frequency response equation: e.g. G(jw)

→ the Laplace transform gives you the frequency response

# Bode plots

*Bode plots illustrate the transfer function/frequency response of a system*

→ Describe the gain and phase shift for the spectrum of all input frequencies in two separate plots



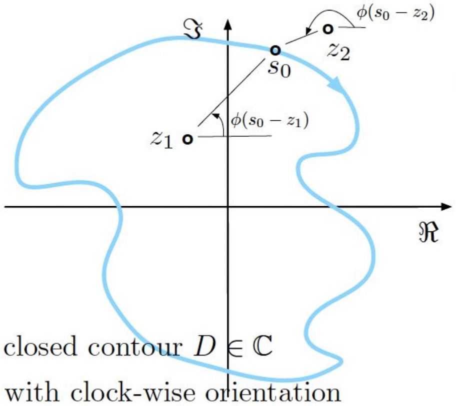
# Nyquist plot

*Nyquist plots **also** illustrate the transfer function/frequency response of a system*

→ Describe the gain and phase shift for the spectrum of all input frequencies in a *single* plot

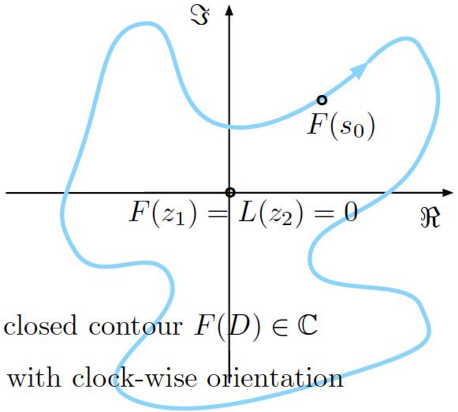
# Cauchy's Argument Principle

s-plane




$$F(s) = \frac{\Pi_{i=1}^{n_z}(s - z_i)}{\Pi_{i=1}^{n_p}(s - p_i)}$$

w-plane

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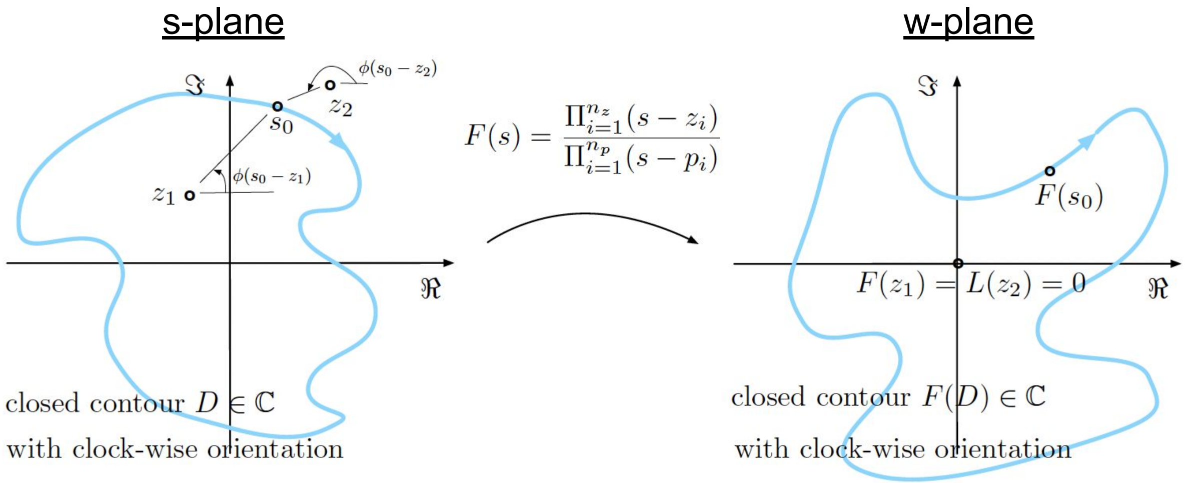


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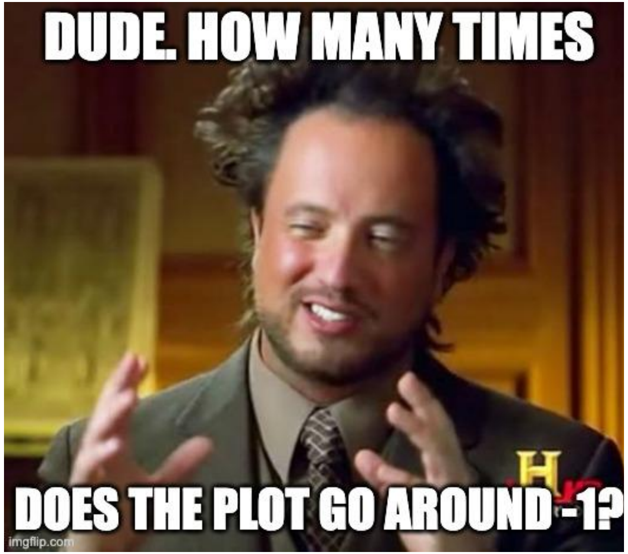

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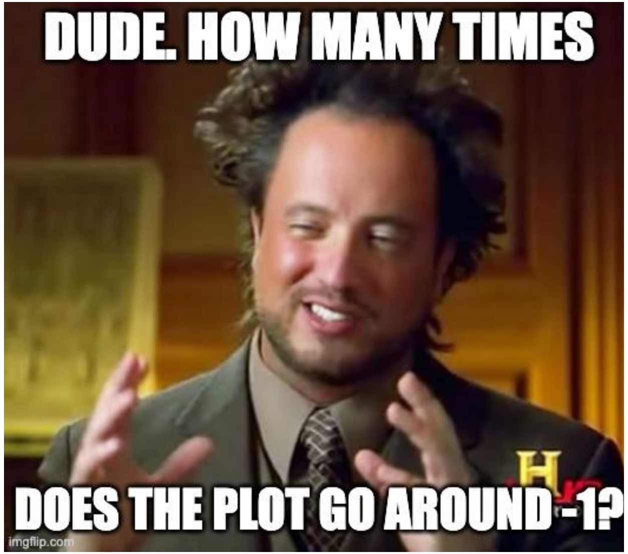


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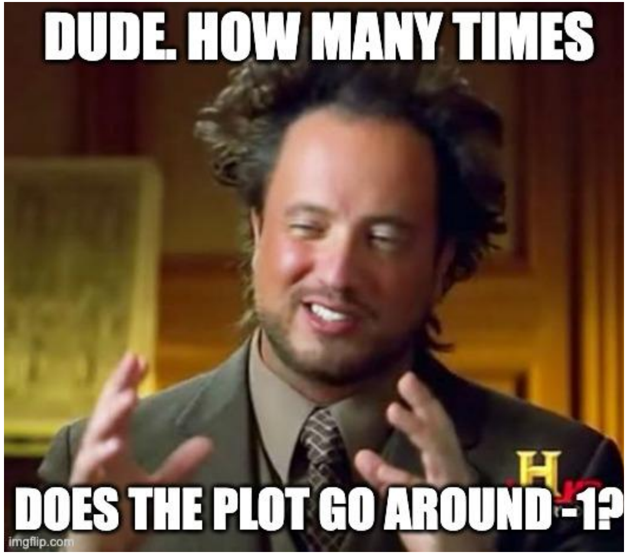


WHAT PLOT, DUDE??
WHY DO WE CARE ABOUT -1???

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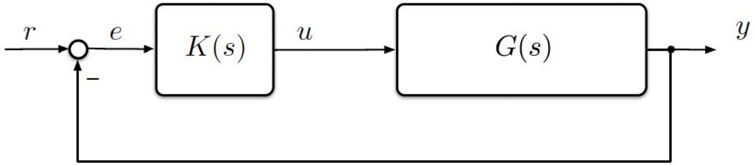


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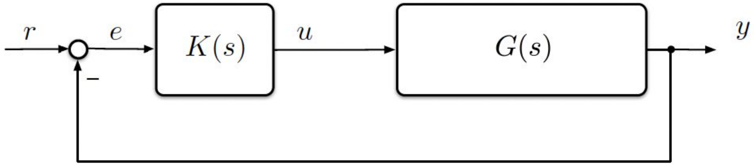


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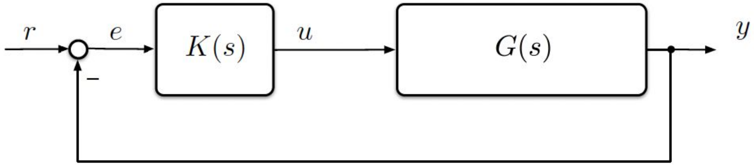


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These poles must be in the LHP

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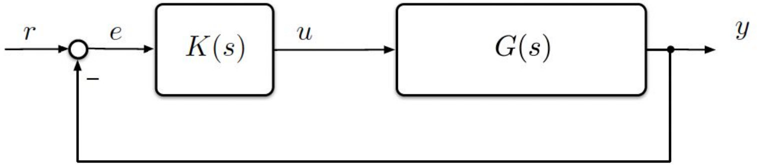


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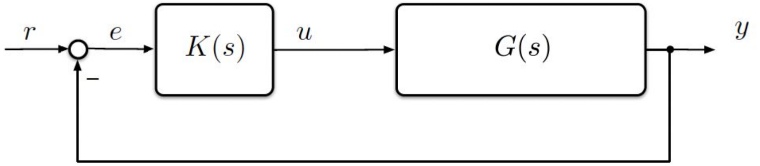


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This is hard to determine from inspection of GK

The zeros of 1+GK must be in the LHP

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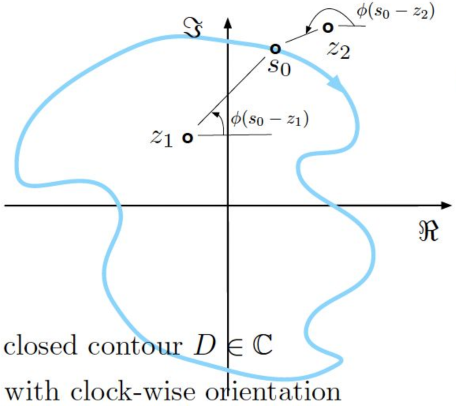
# Cauchy's Argument Principle

*Tells you how many more poles than zeros are in a s-plane loop,*
*based on how many times the w-plane loop encircles the origin in the counter-clockwise (CCW) direction.*

s-plane

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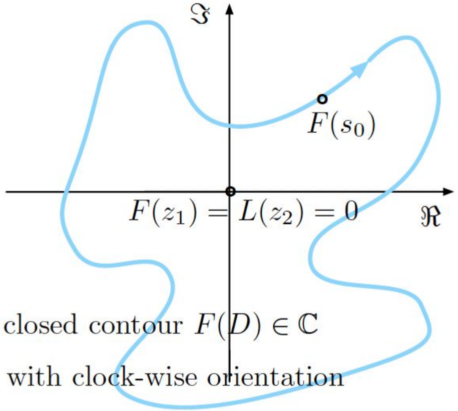


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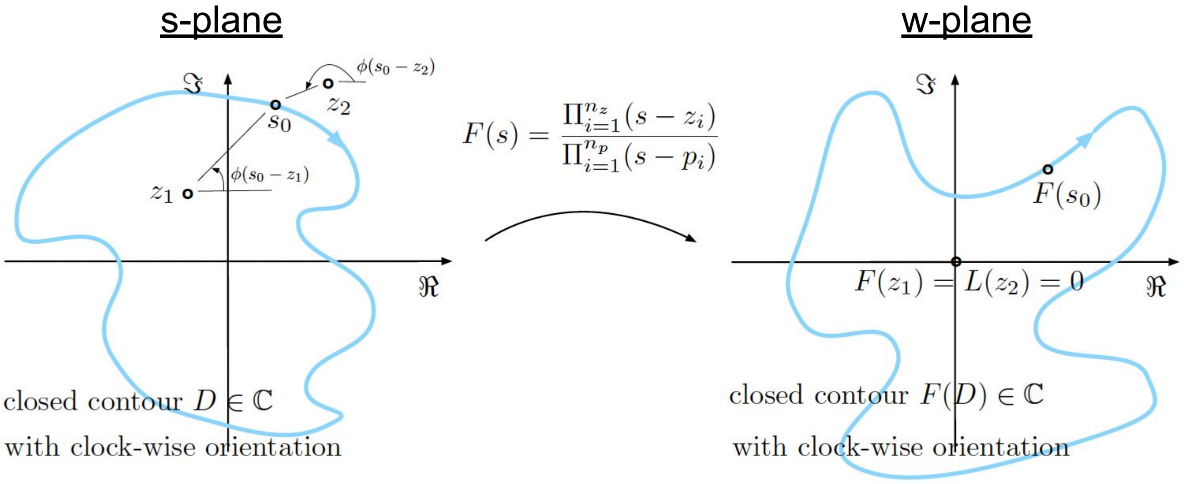


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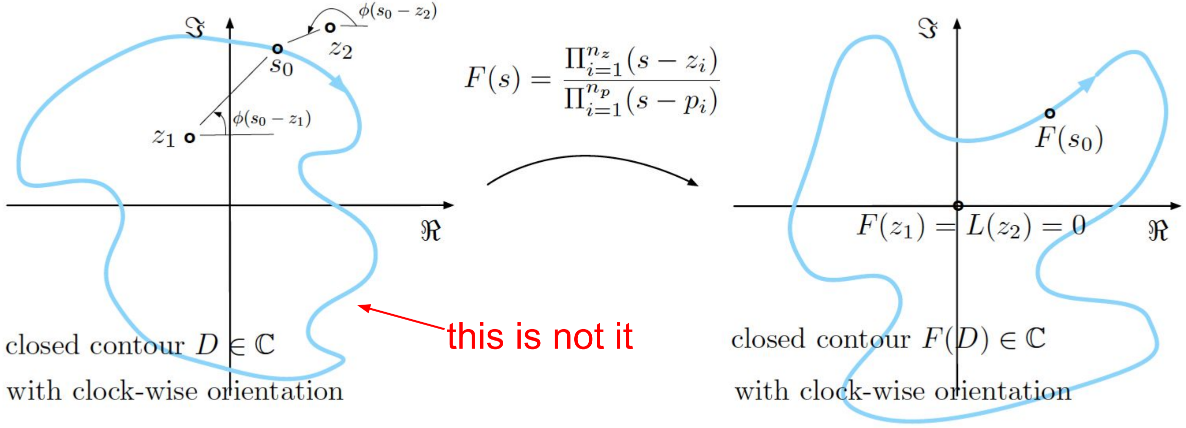
<u>s-plane</u> <u>w-plane</u>

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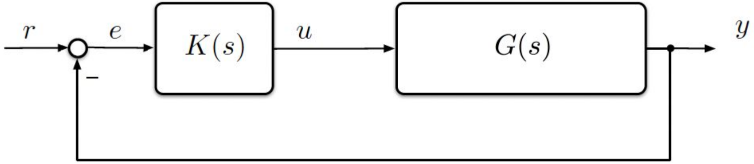
# Nyquist Criterion

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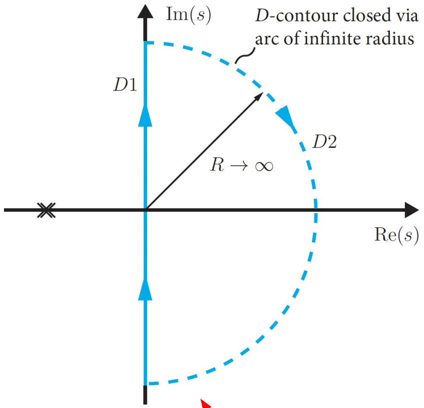


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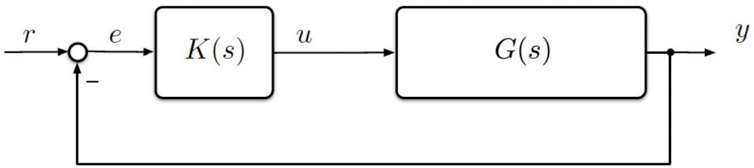


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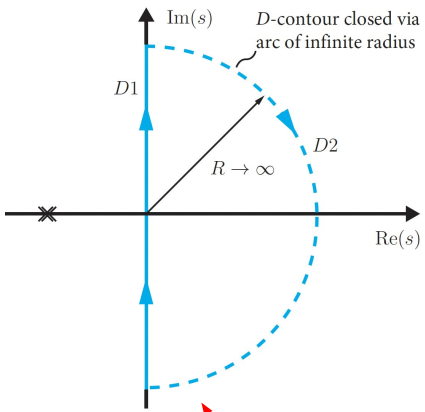


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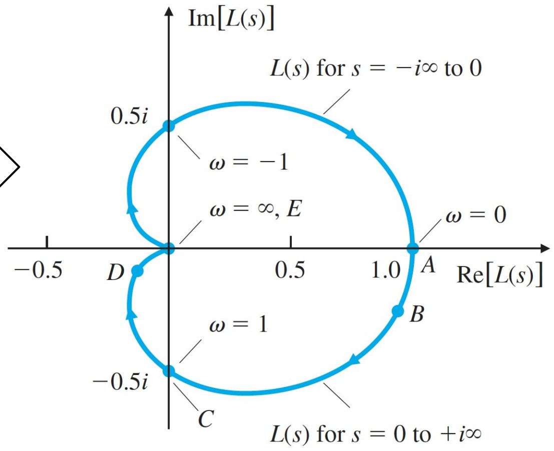


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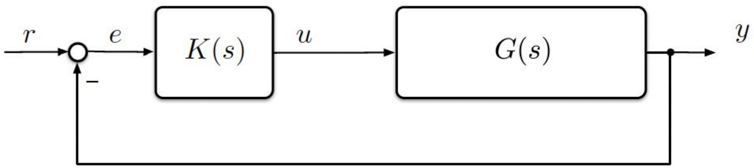


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We are interested in the zeros of 1 + GK
→ encirclements of -1, not 0

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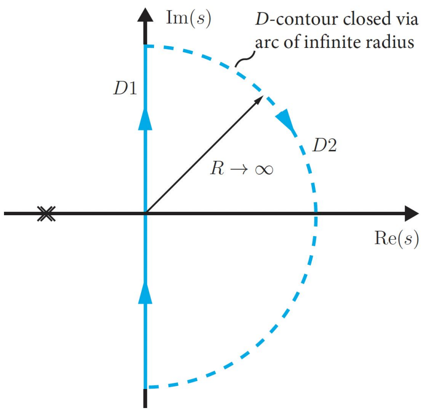


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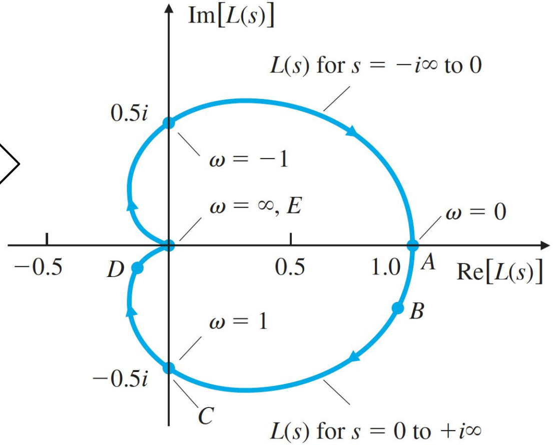

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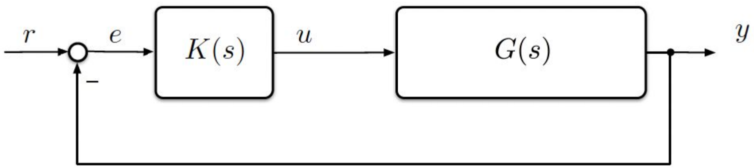


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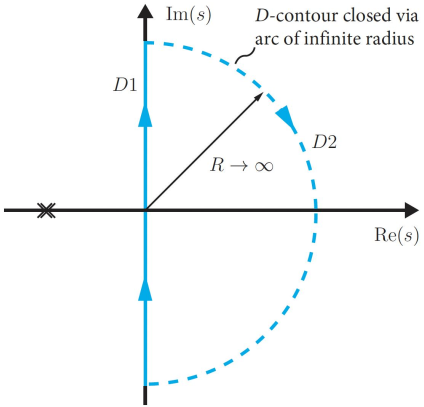


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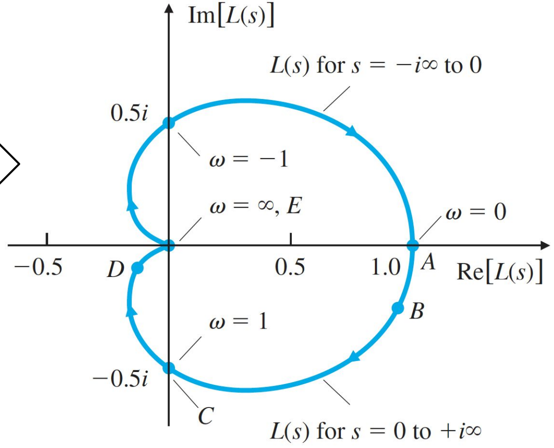


$\Rightarrow$ the closed-loop system is stable if and only if

$$\boxed{N + P = 0}$$

- N is the number of clockwise (CW) encirclements of -1
- P is the number of open loop poles in the RHP

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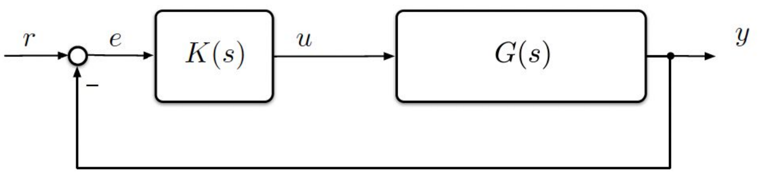


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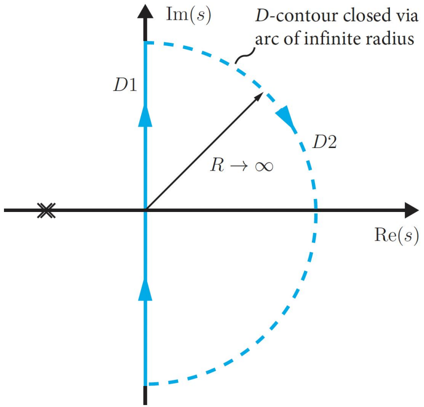


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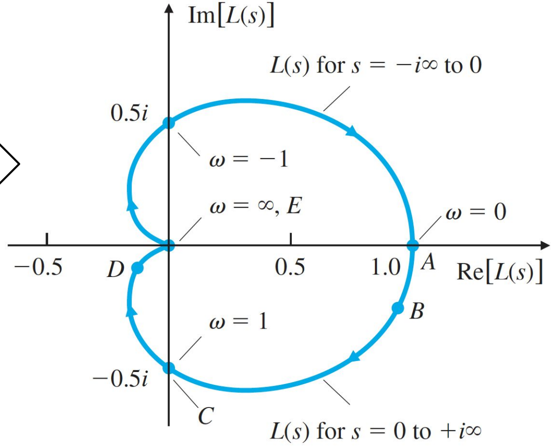


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- N is the number of clockwise (CW) encirclements of -1
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$\Downarrow$

Need 1 CCW encirclement of -1 for each pole of GK in the RHP

# Nyquist Criterion Implementation

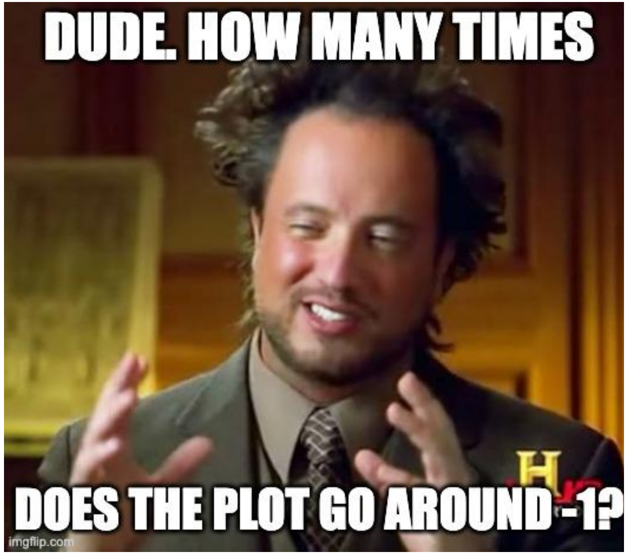
1. Determine the number of OL poles in the RHP (the poles of GK are the poles of 1 + GK)

2. Draw/use python to display the Nyquist Diagram of GK

3. Count net CCW encirclements of -1
   (a CW encirclement cancels out a CCW encirclement)

   → If the net CCW encirclements of -1 = # OL poles in the RHP, the close loop system is stable.

# Why do we care?




WHAT PLOT, DUDE??
WHY DO WE CARE ABOUT -1???

THE W-PLANE CORRESPONDING TO THE D-CONTOUR.
-1 ENCIRCLEMENT TELL US ABOUT CL STABILITY!

# Nyquist Plot, poles on imag. axis

What if there are poles on the imaginary axis?

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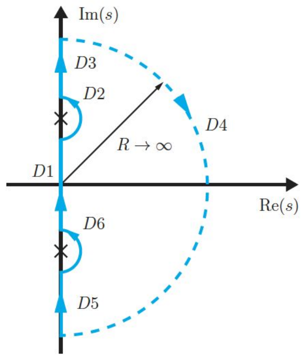



The detour arcs D2 and D6 are mapped to half-circles in the w-plane with infinity radius

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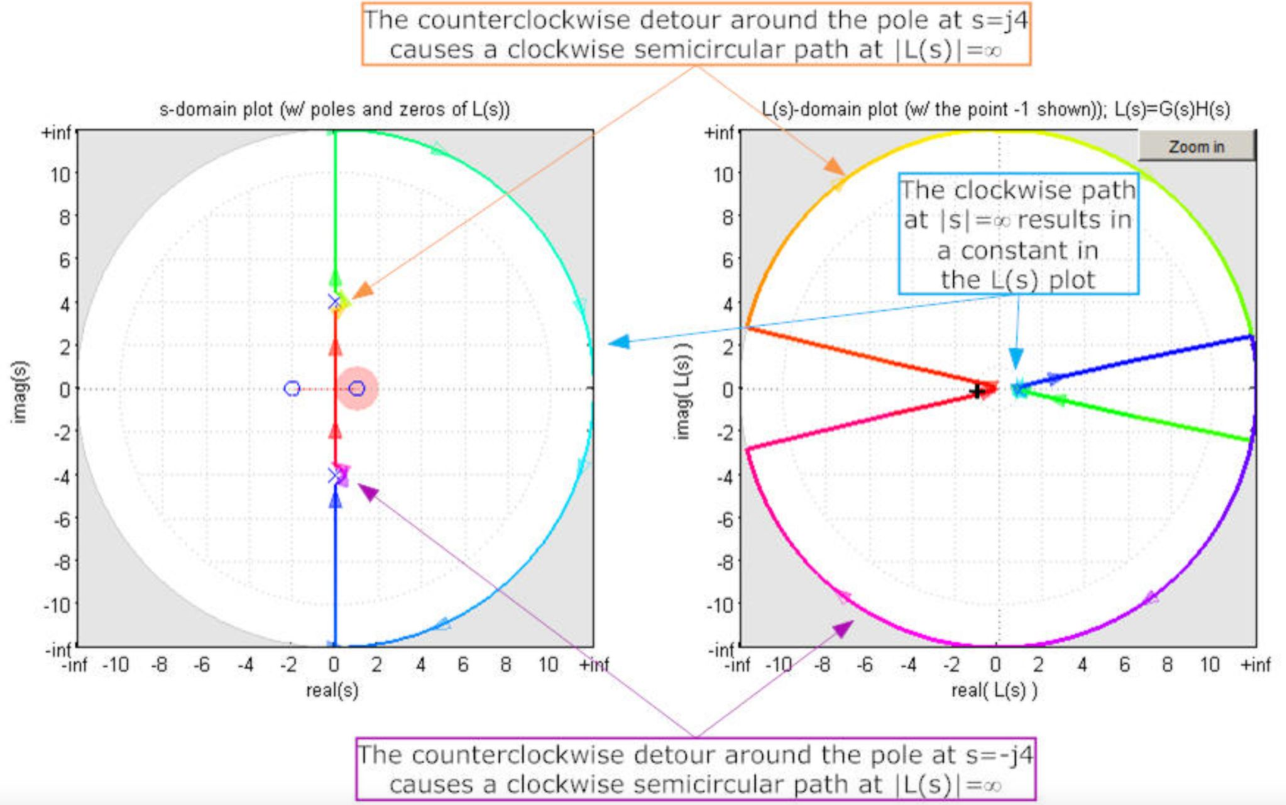
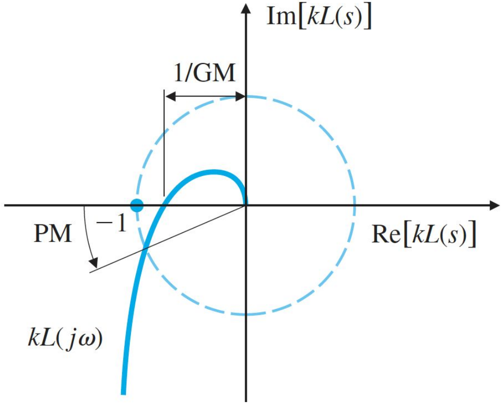



Source: https://lpsa.swarthmore.edu/Nyquist/NyquistStability.html

# Gain and phase margins, Nyquist

<u>Nyquist</u>




- phase margin

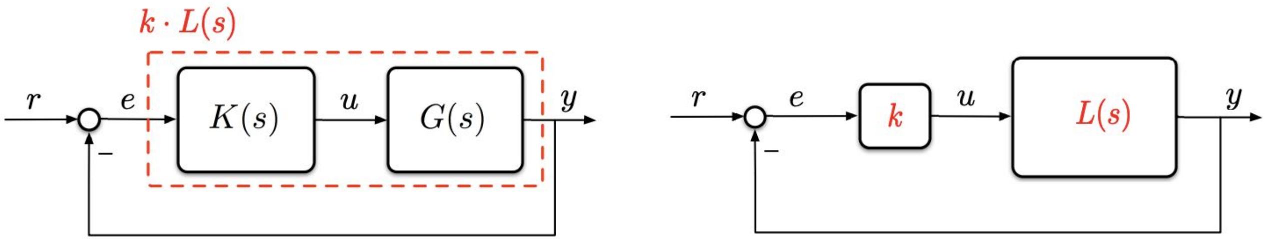
≈ robustness against time delays

- gain margin

≈ robustness against uncertainty in gain

are two measurements of robustness.

# Nyquist Criterion, admissible gains



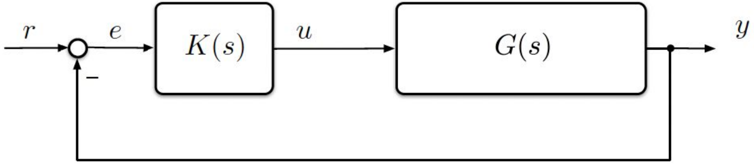

shifted & scaled map of $D$ via $L(s) = \frac{F(s)}{k} - \frac{1}{k}$

$=$ map of $D$ via $F(s)$ shifted by $(-1/k, 0)$ and scaled by $1/k$

$\boxed{\text{count encirclements of } -1/k \text{ via } L(s)}$

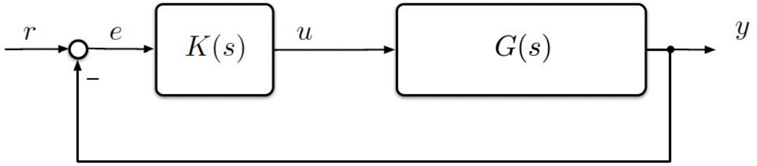
useful to find range of admissible control gains $k$

# Bode Criterion




*Unlike the Nyquist Criterion, the Bode criterion applies only if the OL does not have RHP poles*

# Classical control




Known: G(s)

Desired: Good closed loop behavior (robust, etc.)

Approach: Choose K(s) with as few calculations as possible

robust

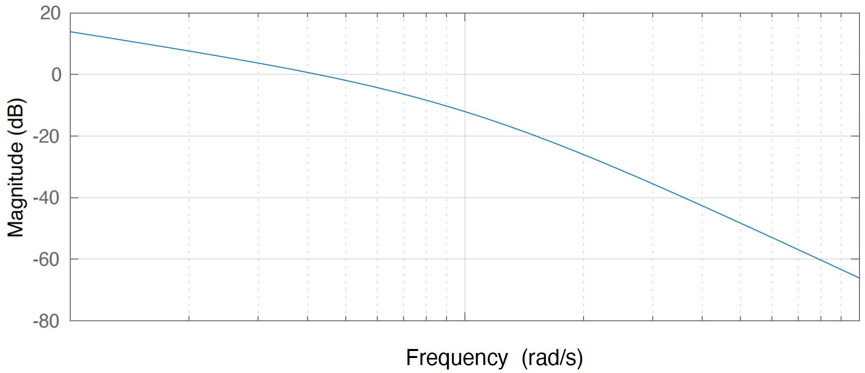
We are interested in closed loop stability, but we design the open loop transfer function.

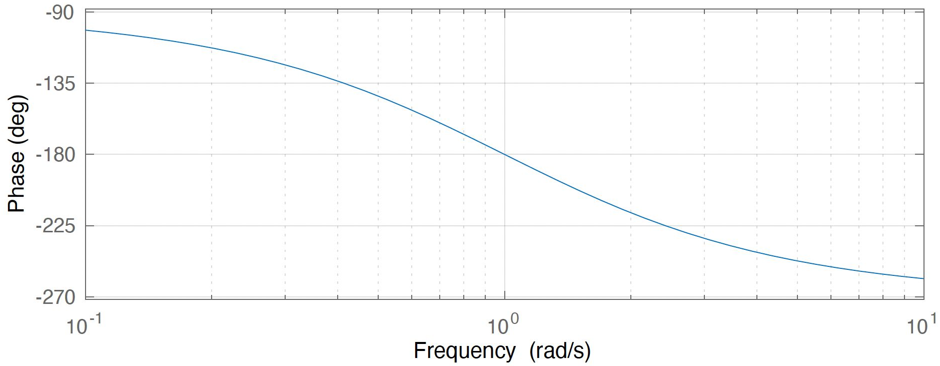
$$\frac{y(s)}{r(s)} = \frac{GK}{1+GK}$$

# Bode plots

*Bode plots illustrate the transfer function/frequency response of a system*

→ Describe the gain and phase shift for the spectrum of all input frequencies



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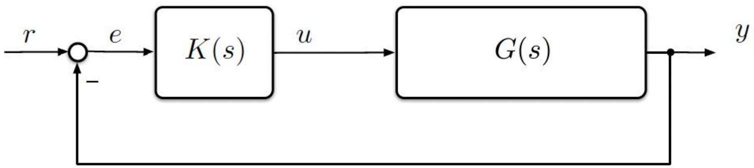


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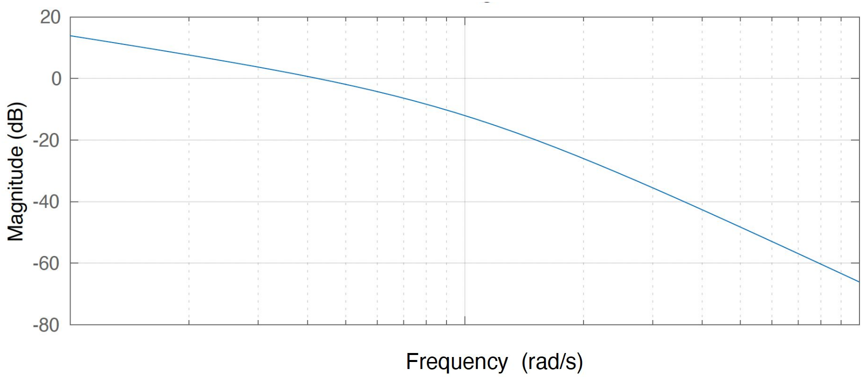
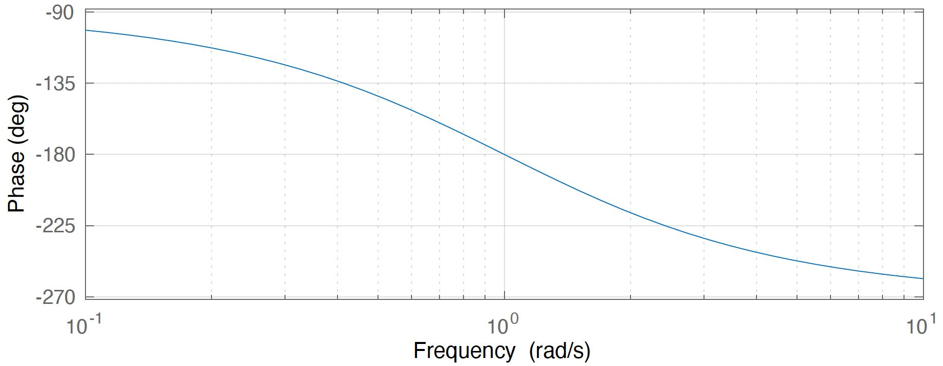
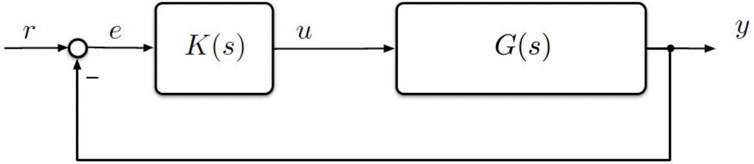



Can we determine the stability of the closed loop system from the open loop Bode plot?

$$\frac{y(s)}{r(s)} = \frac{GK}{1+GK}$$

# Bode Criterion



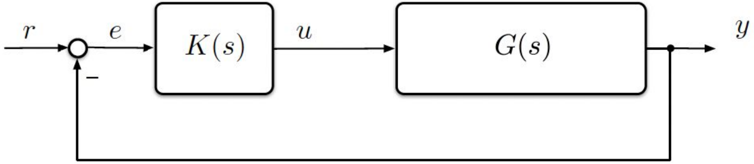

*Bode plots illustrate the transfer function/frequency response of a system*

→ Describe the gain and phase shift for the spectrum of all input frequencies

From Nyquist Criterion:
*If the OL does not have RHP poles*, then the CL is stable when the Nyquist diagram does not encircle -1

→ guaranteed if |GK| < 1 when the phase crosses -180° (and |GK| stays less than 1)

# Bode Criterion

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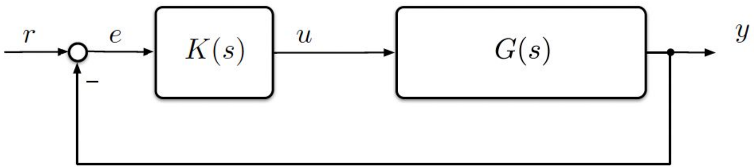
Bode Criterion (applies to open-loop stable/no OL RHP pole systems):

→ *The CL system is stable if the gain is less than 1 (0 dB) when the phase crosses -180°*

# Bode Criterion

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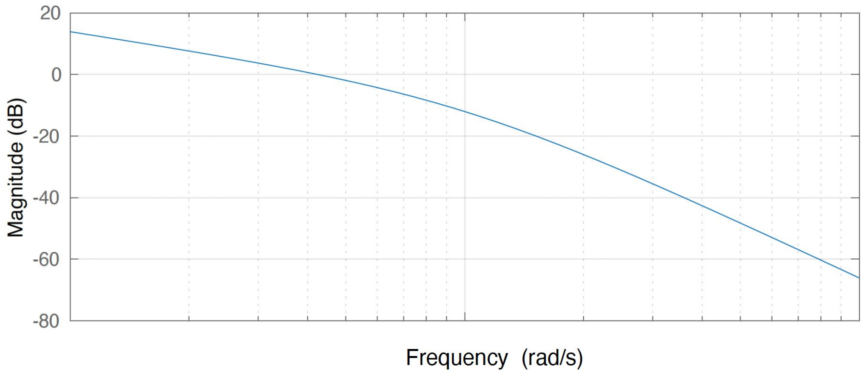
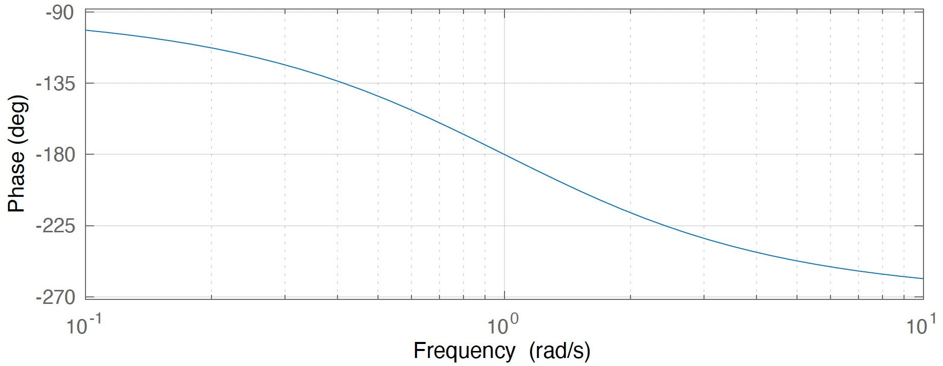


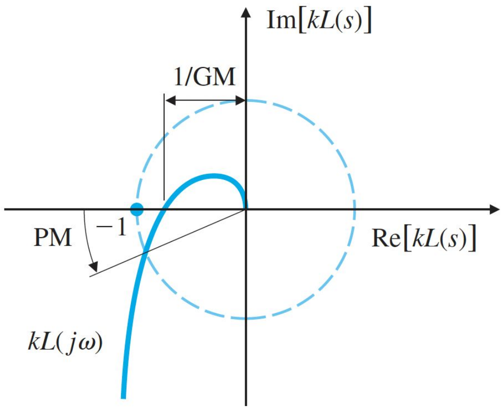




Bode Criterion (applies to open-loop stable/no OL RHP pole systems):

→ *The CL system is stable if the gain is less than 1 (0 dB) when the phase crosses -180°*

# Gain and phase margins

Nyquist



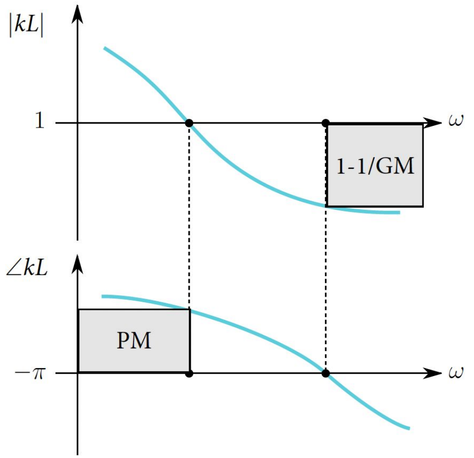

- phase margin

≈ robustness against time delays

- gain margin

≈ robustness against uncertainty in gain

Bode




are two measurements of robustness.

# Why do we care?