Review:
Responses to LTI Systems
& BIBO Stability

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Not BIBO-stable:

Frequency domain representation

Asserts that the input is a sinusoid at frequency w

Output: (also a sinusoid because the system is LTI)

$$y(t) = |G(i\omega)|e^{i(\omega t + \arg(G(i\omega))}$$

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Why frequency-domain?

How to get the frequency domain representation?

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Why frequency-domain?

Because matrix exponentials are tedious and the frequency domain plots have a nice standard form
(Note: *any* signal can be represented by a sum of sinusoidal waves of varying frequency and amplitude)

How to get the frequency domain representation?

Calculate the transfer function and set s = jw

Note:
Transfer functions and frequency domain analysis are typically applied to SISO systems.
They can be applied to MIMO systems as well, but each input-output pair requires its own transfer function

Bode plots

Bode:

Bode plots illustrate transfer functions

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Why is the plot in log-log axes?

Why are Bode plots plotted in “decibels” (i.e. scaled by 20)?

Helpful resource: https://lpsa.swarthmore.edu/Bode/BodeHow.html

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Signal magnitude factors are additive in the log scale: $\lg(a \cdot b) = \lg a + \lg b$

Why are Bode plots plotted in “decibels” (i.e. scaled by 20)?

Because that’s how they started doing it ~100 years ago

$$: \text{'bel'} = \text{signal energy in log-scale}$$

$$|G|_{\text{dB}} = 10 \lg |G|^2 = 20 \lg |G|$$

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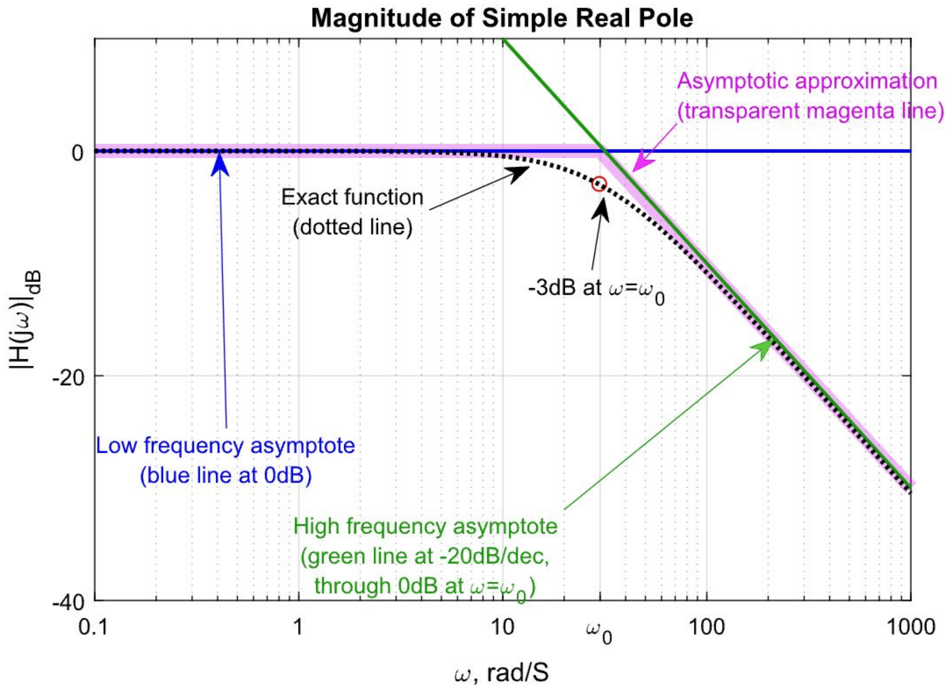
Less-helpful resource:
https://www.youtube.com/watch?v=QHPRrUTn5vQ&ab_channel=CarlosBodefan

Bode:

Bode skiing:

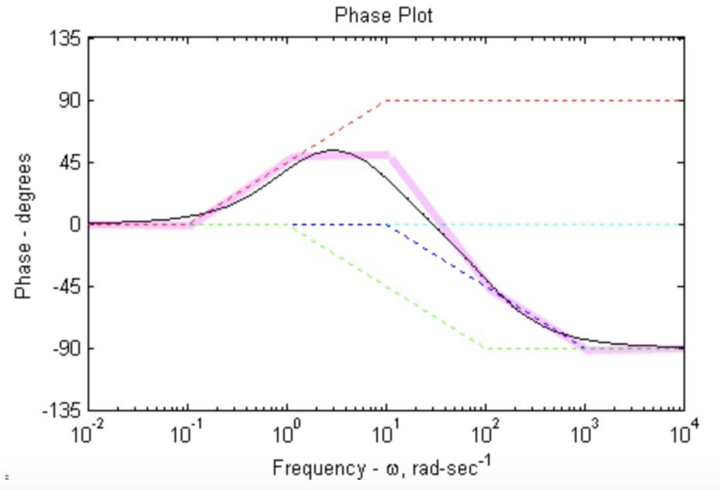
Bode plots

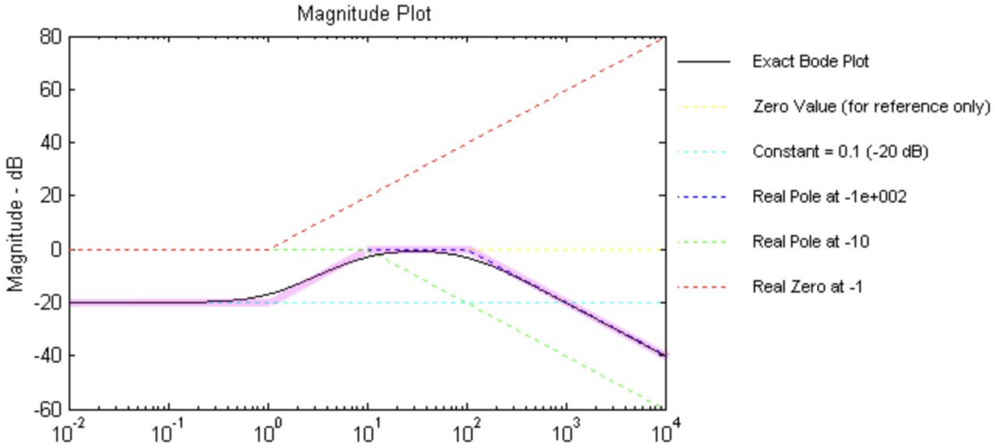


Magnitude of Simple Real Pole
Asymptotic approximation
(transparent magenta line)
Exact function
(dotted line)
-3dB at ω=ω0
Low frequency asymptote
(blue line at 0dB)
High frequency asymptote
(green line at -20dB/dec,
through 0dB at ω=ω0)
|H(jω)|dB
0
-20
-40
0.1
1
10
ω0
100
1000
ω, rad/S

Bode plots example

$$H(s) = 100\frac{(s+1)}{(s+10)(s+100)}$$





Poles and Zeros

$$G(s) = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} + D = K \cdot \frac{\Pi_{i=1}^{m}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)}$$

BIBO stability

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

BIBO stability: $y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau$ is bounded if u(t) is bounded

- **internally stable** *if all eigenvalues of* A *are in the open left half plane; and*
- **BIBO stable** *if all poles of* $G(s) = C(sI - A)^{-1}B + D$ *are in the open left half plane.*

- *internal stability* $\implies$ *BIBO*
- *BIBO* $\implies$ *internal stability if there are no pole/zero cancellations*

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- *internal stability* $\implies$ *BIBO*
- *BIBO* $\implies$ *internal stability if there are no pole/zero cancellations*

If there are pole/zero cancellations, then an unstable internal mode could be excited by the initial condition, even though the unstable mode is not excited by the input or seen at the output

Pole-zero cancellations

$$Y(s) = \underbrace{K \cdot \frac{\Pi_{i=1}^{m}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)}}_{=G(s)} \cdot U(s) = \underbrace{K \cdot \frac{(s - z_1) \cdot (s - z_2) \cdots (s - z_m)}{(s - p_1) \cdot (s - p_2) \cdots (s - p_n)}}_{=G(s)} U(s)$$

Determining BIBO Stability

A system is BIBO stable if:

- the system is internally stable
  - So if have an A-matrix, you can state BIBO stability from the eigenvalues *in the affirmative case*

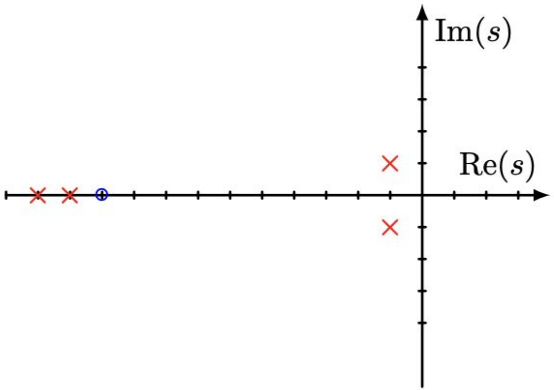
- the transfer function poles are in the LHP
  - Can use the transfer function after pole-zero cancellations

- The integral of impulse response magnitude is bounded: $$\int_{-\infty}^{\infty} |g(t)|dt < \infty \ \ (= \infty)$$
  - This works even when the transfer function is not rational

BIBO stability application

Ignoring fast dynamics:

$$G(s) = \frac{s+10}{(s+11)(s+12)(s^2+2s+2)}$$
$$= \underbrace{\frac{s+10}{(s+11)(s+12)}}_{G_{\text{fast}}(s)} \cdot \underbrace{\frac{1}{s^2+2s+2}}_{G_{\text{slow}}(s)}$$

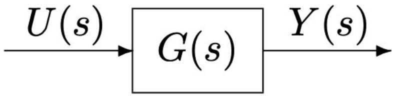


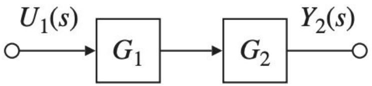
$$\widehat{G}(s) \approx G_{\text{fast}}(0) \cdot G_{\text{slow}}(s)$$

You can only do this if the fast dynamics are BIBO stable!

Block diagram modeling

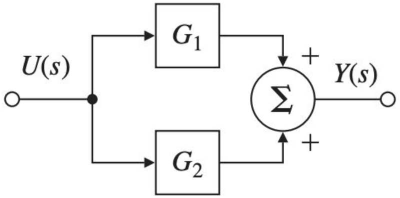
Blocks are transfer functions:





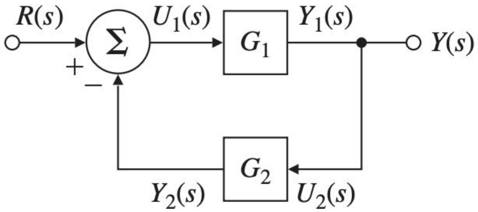
$$\frac{Y_2(s)}{U_1(s)} = G_1 G_2$$

(a) series/cascade



$$\frac{Y(s)}{U(s)} = G_1 + G_2$$

(b) parallel



$$\frac{Y(s)}{R(s)} = \frac{G_1}{1 + G_1 G_2}$$

(c) feedback

Discrete time systems

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

	Continuous Time Systems	Discrete Time Systems
State space:	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$x[k+1] = Ax[k] + Bu[k]$ $y[k] = Cx[k] + Du[k]$
Time-domain convolution:	$\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau$	$\sum_{\ell=0}^{k-1} CA^{k-\ell-1}Bu[\ell]$
Frequency domain:	Laplace transform	z-transform ($z = e^{sT}$)
Transfer function:	$G(s) = C(sI - A)^{-1}B + D$	$G[z] = C(zI - A)^{-1}B + D$
Stability condition:	Poles in left-half plane	Poles in unit-circle
DC gain:	$G(0)$	$G(1)$
Bode plots:	Common and useful	Less common, distorted

Continuous → Discrete time systems

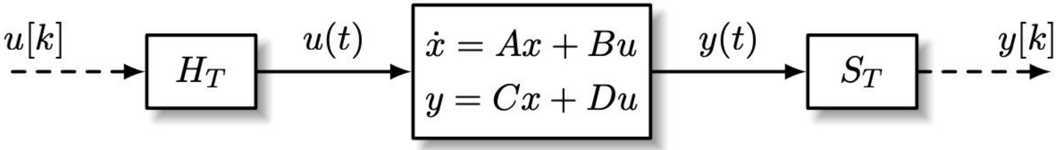
$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

Discretization interval: length of time between discrete system measurements (and actuation), e.g. T

"**0th-order hold**": sample input at time t and use that from time (t) to time (t +T)

0th-order Hold Discretization

Discrete inputs interacting with a continuous system:



For general H(t):

$$x(t_{k+1}) = \underbrace{e^{A(t_{k+1}-t_k)}}_{=e^{AT}} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B u(\tau)\, \mathrm{d}\tau$$

If we keep u(t) constant for each interval:

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)} x(t_k) + \left(\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B\, \mathrm{d}\tau \right) u(t_k)$$

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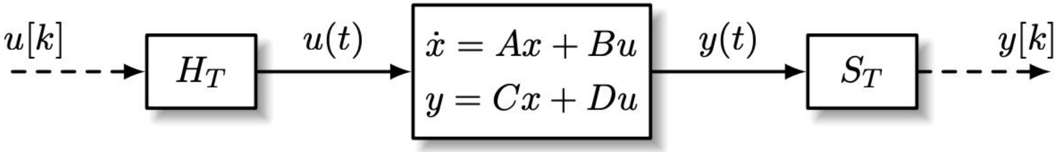

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If we keep u(t) constant for each interval:

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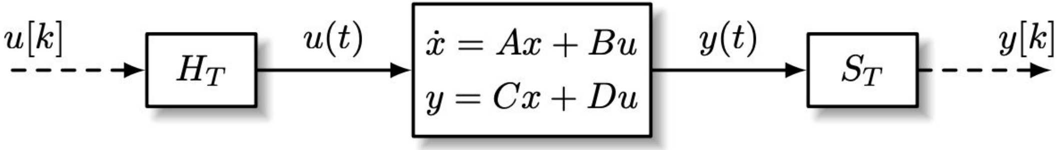

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$\Rightarrow$

$$x[k+1] = A_\mathrm{d}x[k] + B_\mathrm{d}u[k]$$
$$y[k] = Cx[k] + Du[k]$$

Theory: $A_\mathrm{d} = e^{AT}$ $\quad B_\mathrm{d} = \int_0^T e^{A\sigma} B \, \mathrm{d}\sigma = A^{-1}(e^{AT} - I_n)B$ if A is invertible

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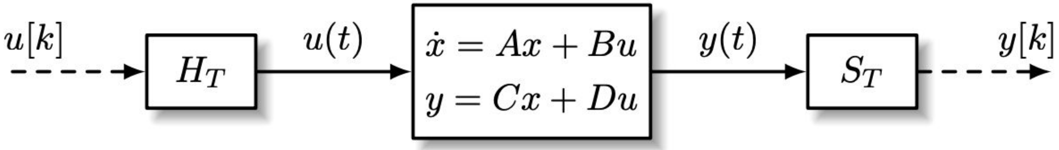

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$$\Longrightarrow$$

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In practice: `sys_ss_dt = control.StateSpace.sample(sys_ss_ct, Ts, method='zoh')`

Bode:

Bode skiing:

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Less-helpful resource:
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