Review:
Solutions to LTI Systems
& Internal Stability

Dr. Keith Moffat

LTI Systems

$$\dot{x}(t) = Ax(t)\,,\; x(0) = x_0$$

Solutions to diagonalizable LTI systems

Modal decomposition

Consider the linear homogeneous system

$$\dot{x} = Ax\,,$$
$$x(0) = x_0\,.$$

Assume that all eigenvectors are **linearly independent** such that A can be diagonalized as

$$A = \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}}_{=V} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}}_{=\Lambda} \underbrace{\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}}_{=W=V^{-1}} \quad \Leftrightarrow \quad WAV = \Lambda\,,$$

where the triple (λ_i, v_i, w_i) denotes an eigenvalue and associated right (respectively, left) eigenvectors of A.

Coordinate transformation: Now let $x = Vz \Leftrightarrow z = V^{-1}x = Wx$:

$$\frac{d}{dt}Vz = V\dot{z} = AVz \quad \Leftrightarrow \quad \dot{z} = WAVz = \Lambda z$$
$$z(0) = Wx(0) = Wx_0\,.$$

In these coordinates, the system is diagonal, i.e., all components are decoupled:

$$\dot{z}_i = \lambda_i z_i\,, \qquad i \in \{1, \dots, n\}\,.$$

This system representation is called the **modal form**. Its solution is given by

$$z_i(t) = e^{\lambda_i t} z_i(0)\,, \qquad i \in \{1, \dots, n\}\,.$$

Solutions to diagonalizable LTI systems

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Under special conditions

$$A = \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}}_{=V} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}}_{=\Lambda} \underbrace{\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}}_{=W=V^{-1}} \quad \Leftrightarrow \quad WAV = \Lambda\,,$$

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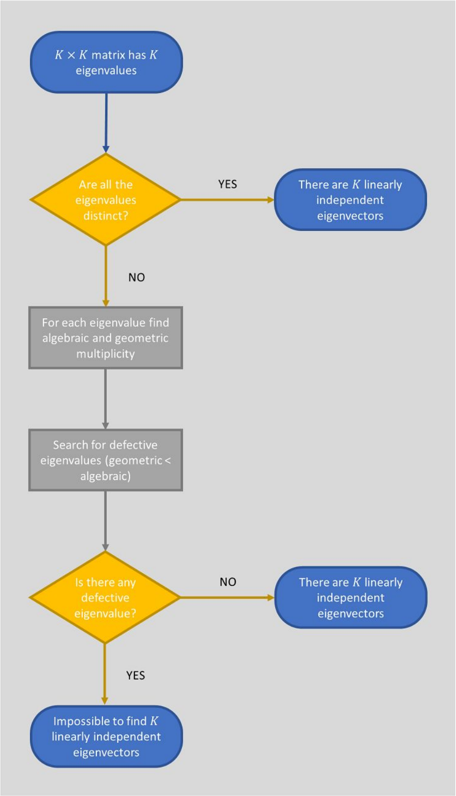
Independent equations for each mode

Solutions to diagonalizable LTI systems

Matrix is diagonalizable ⟺ All the eigenvectors are linearly independent

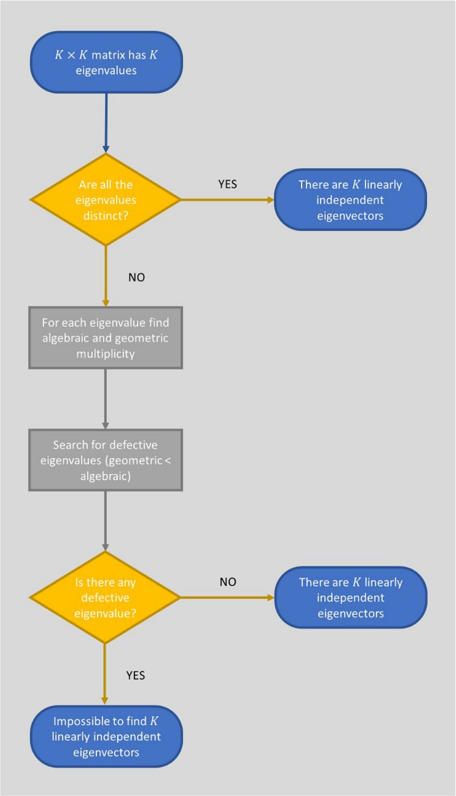


Solutions to diagonalizable LTI systems

Matrix is diagonalizable ⟺ All the eigenvectors are linearly independent

This is true for symmetric-real matrices, but is not always the case.

What do you do when an LTE system's dynamics matrix A is not diagonalizable?



Solutions to general LTI systems

$$\dot{x}(t) = Ax(t)\,,\; x(0) = x_0$$

$$\Downarrow$$

$$x(t) = e^{At}x_0\,.$$

The “matrix exponential” to the rescue!

Solutions to general LTI systems

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The “matrix exponential” to the rescue!

The matrix exponential

Definition: for a square matrix A the *matrix exponential* is the series

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{6} A^3 t^3 + \dots$$

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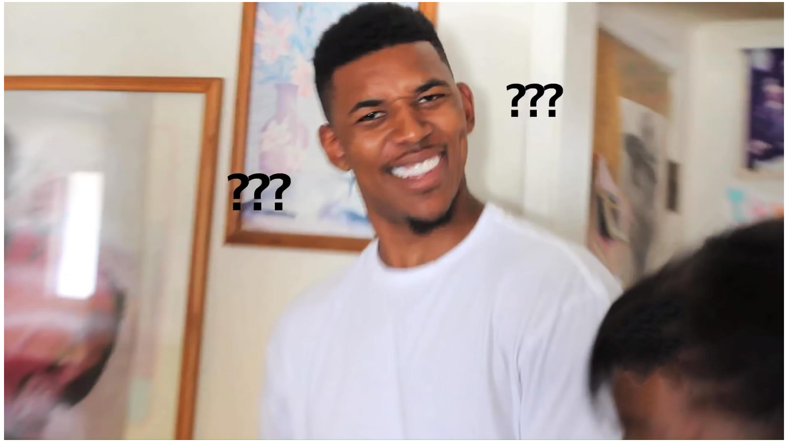
…an infinite series.



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Properties of the matrix exponential

1 derivative: $\frac{d}{dt} e^{At} = A e^{At}$

2 integral: $e^{At} = I + A \int_0^t e^{A\tau}\, d\tau$

3 semi-group: $e^{At} e^{A\tau} = e^{A(t+\tau)}$

4 inverse: $\left(e^{At}\right)^{-1} = e^{-At}$

5 commutativity if A and B commute: $e^{(A+B)t} = e^{At} e^{Bt}$

6 Cayley Hamilton: e^{At} can be expressed as linear combination of $I, A, A^2, \dots, A^{n-1}$ with some coefficients $\mu_0(t), \dots, \mu_{n-1}(t) \in \mathbb{R}$:

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = \mu_0(t) I + \mu_1(t) A + \dots + \mu_{n-1}(t) A^{n-1}$$

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pretty cool… and useful.

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also pretty cool, and sometimes useful.

Solving the matrix exponential

So how do we get $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots$???

Five ways to compute the matrix exponential:

- Series expansion: if the system matrix A is nilpotent, apply the above definition
- Laplace transform: $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$
- Diagonalization: $e^{At} = Ve^{\Lambda t}W = Ve^{\Lambda t}V^{-1}$
- Cayley-Hamilton:

  $$e^{At} = \mu_0(t)I + \mu_1(t)A + \cdots + \mu_{n-1}(t)A^{n-1}$$

  $$\begin{bmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \mu_0(t) \\ \vdots \\ \mu_{n-1}(t) \end{bmatrix}$$

- Educated guess

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Five ways to compute the matrix exponential:

- Series expansion: if the system matrix A is nilpotent, apply the above definition ⇦ Easiest. Doesn't always apply
- Laplace transform: $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$
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- Educated guess ⇦ Easiest. Need a good guess

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⟸ Requires symbolic matrix inversion, difficult for matrices n > 2

– Diagonalization: $e^{At} = V e^{\Lambda t} W = V e^{\Lambda t} V^{-1}$

– Cayley-Hamilton:

$$e^{At} = \mu_0(t) I + \mu_1(t) A + \cdots + \mu_{n-1}(t) A^{n-1}$$

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  ⇦ Useful, requires solving a system of linear equations to get the coefficients

- Educated guess

Cayley-Hamilton

Characteristic polynomial $p(\lambda)$ of the matrix $A \in \mathbb{R}^{n \times n}$:

$$\det(\lambda I - A) = \underbrace{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n}_{= p(\lambda)} .$$

Cayley Hamilton: the matrix A satisfies its characteristic polynomial

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n-1}A^{n-1} + A^n = 0 .$$

Consequence: every matrix power A^k for $k \geq n$ can be expressed as linear combination of $I, A, A^2, \ldots, A^{n-1}$. In particular,

$$A^n = -\left(a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n-1}A^{n-1}\right) .$$

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Derived as a scalar equation

Cayley Hamilton: the matrix A satisfies its characteristic polynomial

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Matrix equation for A

Consequence: every matrix power A^k for $k \geq n$ can be expressed as linear combination of $I, A, A^2, \ldots, A^{n-1}$. In particular,

$$A^n = -\left(a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n-1}A^{n-1}\right) .$$

Does not tell us what the a coefficients are (we have to solve for those)

Continuous LTI System with inputs

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau,$$
$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

Discrete LTI System with inputs

$$x_{k+1} = A_\mathrm{d} x_k + B_\mathrm{d} u_k,$$
$$y_k = C_\mathrm{d} x_k + D_\mathrm{d} u_k.$$

Discretization of LTI systems with inputs

$$\left[\begin{array}{c|c} A_{\mathrm{d}} & B_{\mathrm{d}} \\ \hline C_{\mathrm{d}} & D_{\mathrm{d}} \end{array}\right] = \left[\begin{array}{c|c} e^{AT} & \int_0^T e^{A\tau}\, d\tau\, B \\ \hline C & D \end{array}\right]$$

(T is the sampling time)

$$\text{if } \det(A) \neq 0,\ B_d = A^{-1}(e^{AT} - I)B$$

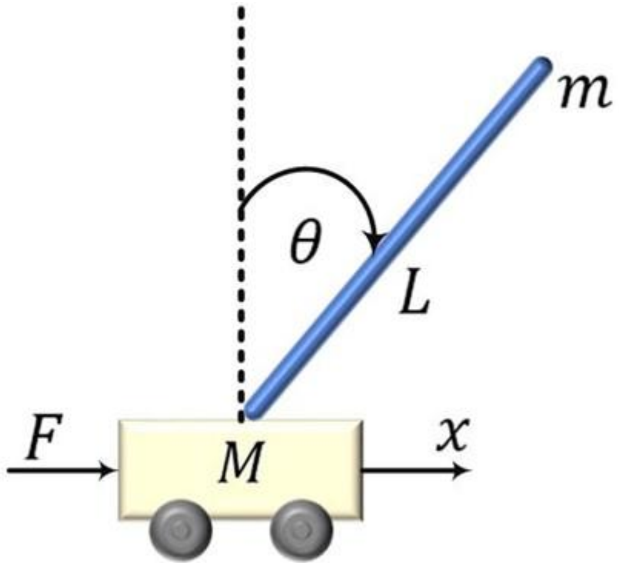
Discrete LTI System with inputs

$$\begin{aligned} x_{k+1} &= A_{\mathrm{d}} x_k + B_{\mathrm{d}} u_k, \\ y_k &= C_{\mathrm{d}} x_k + D_{\mathrm{d}} u_k. \end{aligned}$$

Solution:

$$\begin{aligned} x_k &= A_{\mathrm{d}}^k x_0 + \sum_{\tau=0}^{k-1} A_{\mathrm{d}}^{k-1-\tau} B_{\mathrm{d}} u_\tau, \\ y_k &= C_{\mathrm{d}} A_{\mathrm{d}}^k x_0 + \sum_{\tau=0}^{k-1} C_{\mathrm{d}} A_{\mathrm{d}}^{k-1-\tau} B_{\mathrm{d}} u_\tau + D_{\mathrm{d}} u_k. \end{aligned}$$

Stability



Stability

Stability of continuous LTI systems

Transfer function representation

Consider $Y(s) = \frac{1}{s^2+a_1 s+a_0} U(s)$ or its associated state-space realization

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{=A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=B} u \quad , \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=C} x \, .$$

The characteristic polynomial is given by $p_2(\lambda) = \lambda^2 + a_1\lambda + a_0$.

Routh-Hurwitz criterion for polynomial of degree 2: The roots of $p_2(\lambda)$ have strictly negative real part **if and only if** $a_0 > 0$ and $a_1 > 0$.

Routh-Hurwitz criterion for polynomial of degree n: The roots of

$$p_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

have strictly negative real part **only if** $a_i > 0$ for all $i \in \{0, 1, \ldots, n-1\}$.

Stability of continuous LTI systems

State space representation

The continuous-time linear homogeneous system $\dot{x} = Ax$ is

1. **globally asymptotically stable** if and only if all eigenvalues of A have strictly negative real part $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \text{spectrum}(A)$.

   Such a matrix is also called a **Hurwitz** matrix. We often abbreviate the condition $\{\mathrm{Re}(\lambda) < 0 \ \forall \lambda \in \text{spectrum}(A)\}$ as spectrum(A)$\subset \mathbb{C}_-$.

2. **stable** if and only if all eigenvalues of A have non-positive real part $\mathrm{Re}(\lambda) \leq 0$ for all $\lambda \in \text{spectrum}(A)$, & all Jordan blocks corresponding to eigenvalues with zero real parts are of dimension 1×1.

3. **unstable** if and only if at least one eigenvalue of A has a positive real part or zero real part with corresponding Jordan block larger than 1×1.

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Stability of discrete LTI systems

State space representation

The discrete-time linear homogeneous system $x^+ = Ax$ is

1. **globally asymptotically stable** if and only if all eigenvalues of A have modulus strictly less than one, $|\lambda| < 1$ for all $\lambda \in \text{spectrum}(A)$.

   Such a matrix is also called a **Schur** matrix. We often abbreviate the condition $\{|\lambda| < 1 \ \forall \lambda \in \text{spectrum}(A)\}$ as spectrum(A)$\subset \mathbb{D}$.

2. **stable** if and only if all eigenvalues of A have modulus less than one, $|\lambda| \leq 1$ for all $\lambda \in \text{spectrum}(A)$, & all Jordan blocks corresponding to eigenvalues with modulus one are of dimension 1×1.

3. **unstable** if and only if at least one eigenvalue of A has a modulus larger than one or modulus one with corresponding Jordan block larger than 1×1.

Nonlinear system stability

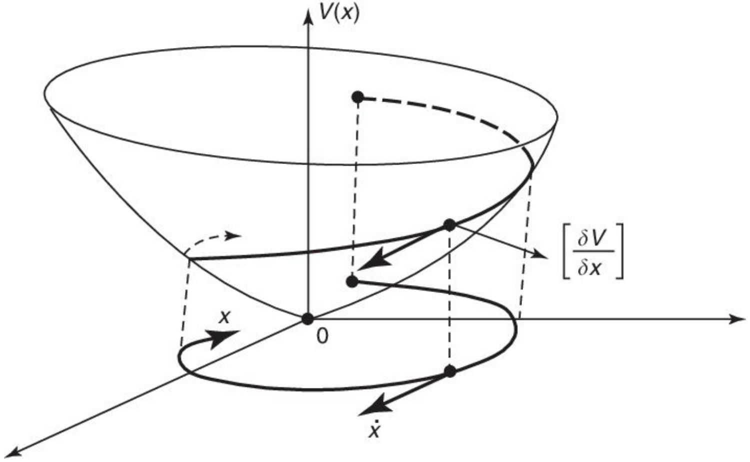
Use Lyapunov:

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \,\forall\, x \in S\backslash\{0\} \quad \text{(positive definite)}$$

and

$$\dot{V}(x) = \nabla V(x)^T \cdot f(x) \leq 0 \quad x \in S \quad \text{(negative semidefinite)}$$

then $x = 0$ is stable.



Nonlinear system stability verification

Consider the system $\dot{x} = f(x)$ with an equilibrium at the origin: $f(0) = 0$

1 **via the solution** (typically numerically):

> $\|x(t)\|$ bounded (& convergent) $\Longleftrightarrow$ (asymptotic) stability

$\Rightarrow$ not practical for state-space dimension $n \geq 2$ (useful in linear case)

2 **via linearization:**

> The origin of $\dot{x} = f(x)$ is locally asymptotically stable (resp., unstable) if the linearized system $\frac{d}{dt}\Delta x = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} \Delta x$ is asymptotically stable (resp., unstable).

$\Rightarrow$ often applicable, but now always; see e.g., $\dot{x} = \pm x^3$


STABILITY
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