Review: Representations

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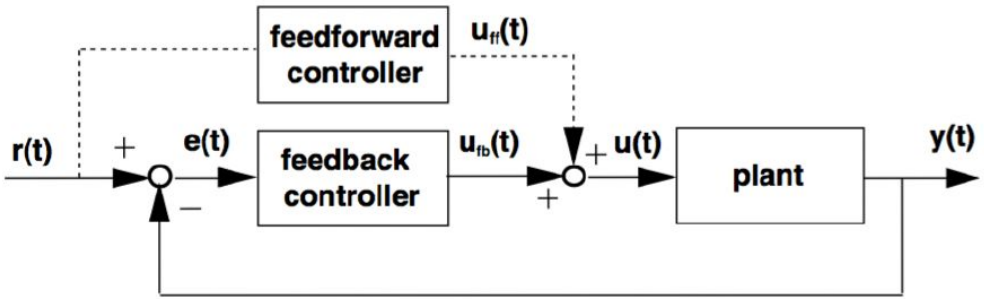
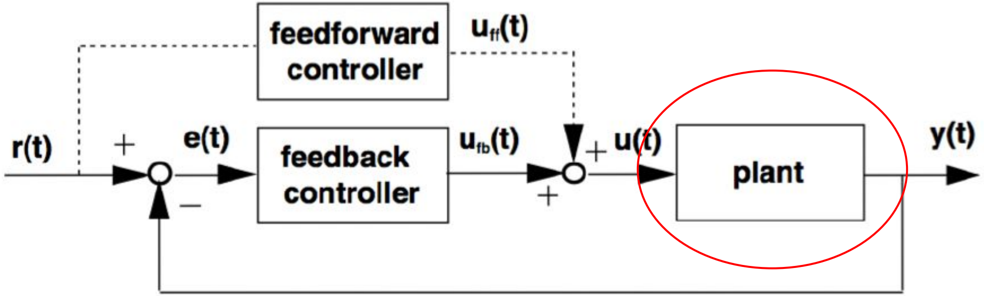
Standard Feedback Control



System Modelling



Laplace Transform:

The (one-sided) **Laplace transform** of a function $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

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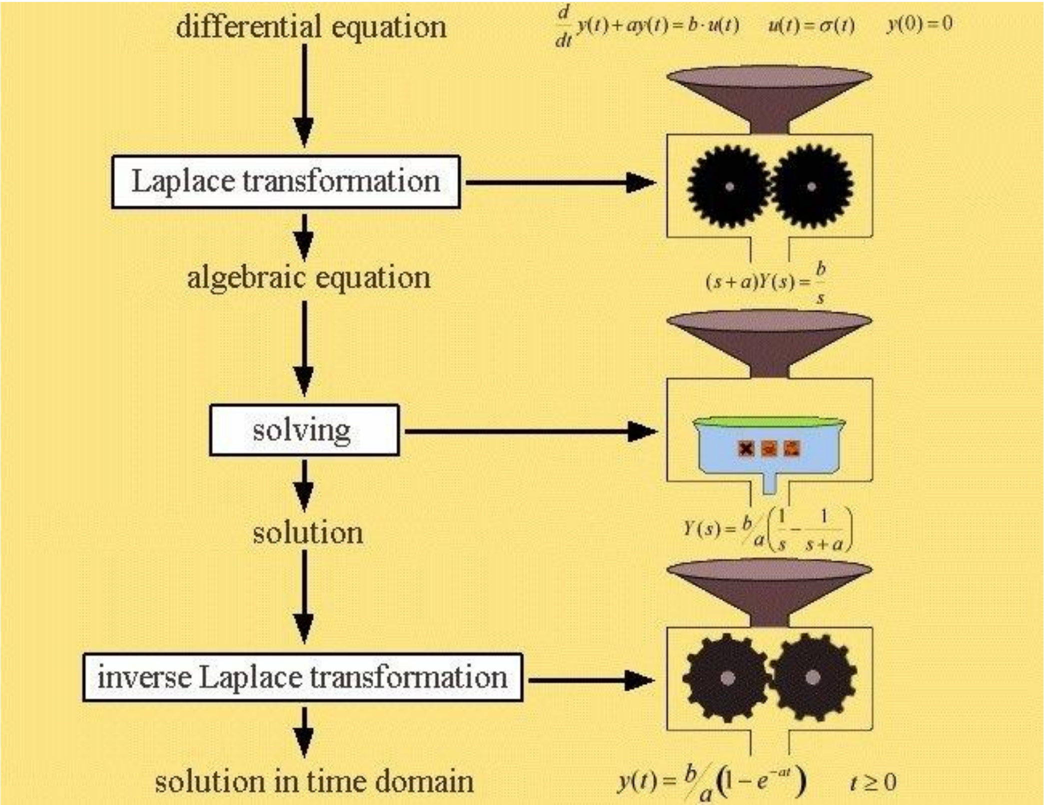

Laplace Transform:

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2 + a^2}$	$s > 0$
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$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2 + b^2}$	$s > a$
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- **linearity**

$$\mathcal{L}\{\alpha_1 f_1 + \alpha_2 f_2\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

- **delay**

$$\mathcal{L}\{f(t-\tau)\} = e^{-\tau s} F(s)$$

- **integral formula**

$$\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \frac{1}{s} F(s)$$

- **derivative formula**

$$\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = sF(s) - f(0)$$

- **convolution**

$$\mathcal{L}\{g * f\} = G(s) \cdot F(s)$$

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Laplace Transform:

Differential equation models of physical systems

The Laplace Transform

Padé approximation for time-delay

$$Y(s) = e^{-sT}U(s) = \frac{e^{-sT/2}}{e^{sT/2}}U(s) = \frac{1 - sT/2 + \dots}{1 + sT/2 + \dots}U(s)$$

$$\approx \frac{2 - sT}{2 + sT}U(s)\,,$$

Final Value Theorem

If a final (finite) value $f(t \to \infty)$ exists, then $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$.

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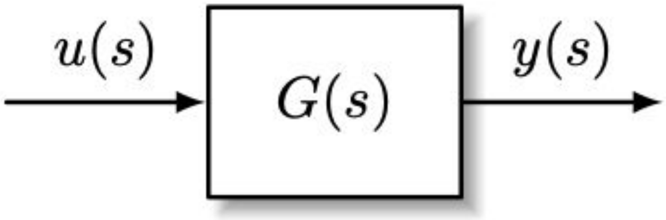
Transfer functions (LTI systems)



$$G(s) = Y(s)/U(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

State space (LTI systems)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du\,, \end{aligned}$$

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du\,, \end{aligned}$$

$$n+q\left\{\underbrace{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}_{n+p}\right.$$

Transfer function → State space

$$Y(s) = (b_2 s^2 + b_1 s + b_0) \underbrace{\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)}_{V(s)}$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & & 0 & 1 \\ -a_0 & -a_1 & \dots & & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{:=A} x + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{:=B} u$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}}_{:=C} x \,.$$

Transfer function ← State space

You can always use the formula $Y(s) = \underbrace{\left(C(sI-A)^{-1}B + D\right)}_{=G(s)} U(s)$

However calculating $(sI-A)^{-1}$ can be difficult when A has more than two dimensions.
In this case, the following canonical forms can sometimes be used:

Controllable Canonical Form:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & & 0 & 1 \\ -a_0 & -a_1 & \dots & & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{:=A} x + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{:=B} u$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}}_{:=C} x\,.$$

$$Y(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} U(s)$$

Observable Canonical Form:

$$\dot{x} = \begin{bmatrix} 0 & \cdots & \cdots & -a_0 \\ 1 & \cdots & 0 & -a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} x$$

$$Y(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} U(s)$$

Nonlinear systems and linearizations

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\Delta x &= \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)}}_{=A} \Delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)}}_{=B} \Delta u \\ y &= \underbrace{\left.\frac{\partial h}{\partial x}\right|_{(x^*,u^*)}}_{=C} \Delta x + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{(x^*,u^*)}}_{=D} \Delta u \,. \end{aligned}$$

Today's exercise session

You  The Laplace Transform